Complete Analytic Calculation of Drell-Yan Angular Coefficients at $\mathcal{O}(\alpha_s^2)$

MSc presentation

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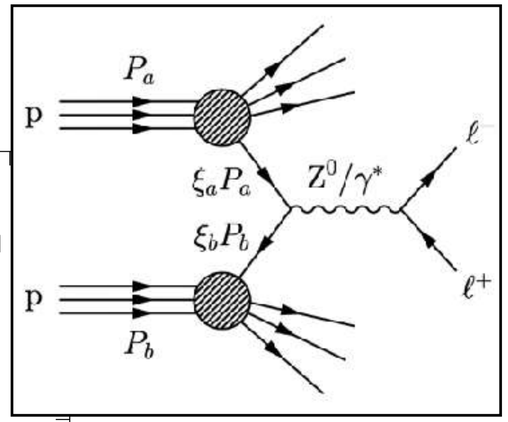
04.07.2025

Roadmap

- **Introduction:** Drell Yan, q_T - spectrum, angular coefficients, radiative corrections
- **How to get the angular coefficients**
  - Pole structure, PDF renormalization
  - V+jet threshold factorization as an analytic subtraction
    - Spinor helicity formalism for loop contributions
    - Main pipeline for real terms
  - Flavor decomposition
- **Conclusion**

Drell-Yan

Transverse momentum spectrum



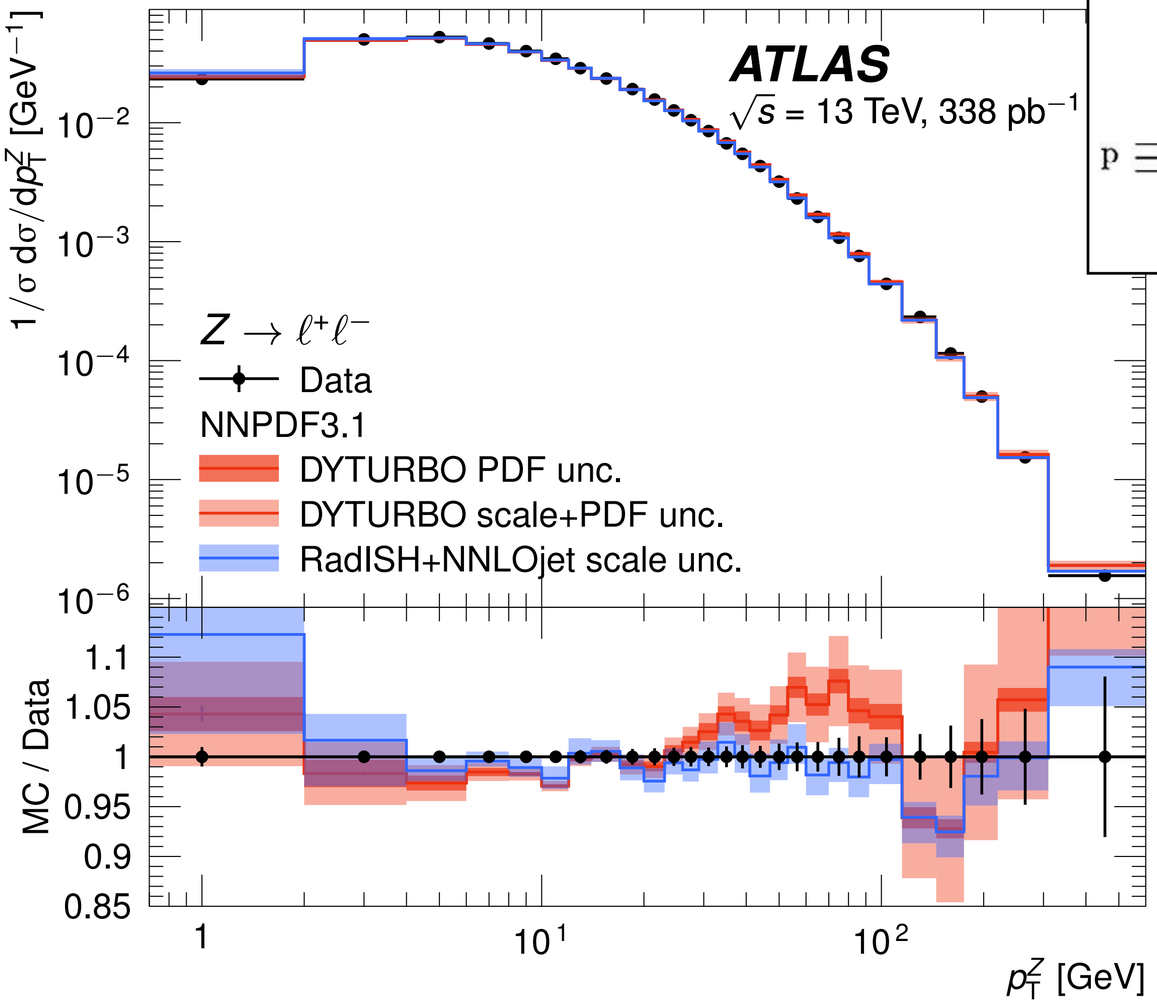
$$p(P_a)\,p(P_b) \to V(q)\,X \quad \text{with}$$

$$Z/\gamma^*(q) \to \ell^-(p_1)\,\ell^+(p_2)$$

$$Z \to \nu(p_1)\,\bar{\nu}(p_2)$$

$$W^+(q) \to \nu_\ell(p_1)\,\ell^+(p_2)$$

$$W^-(q) \to \ell^-(p_1)\,\bar{\nu}_\ell(p_2)$$



- One of the most precisely measured observables at the LHC [ATLAS '24]
- Has been used to obtain:

  W boson mass — m_W [CMS '24]

  Strong coupling — $\alpha_s(m_Z)$

  [S.Camarda, G.Ferrera, M.Schott '22]

  [ATLAS'24]

Angular coefficients

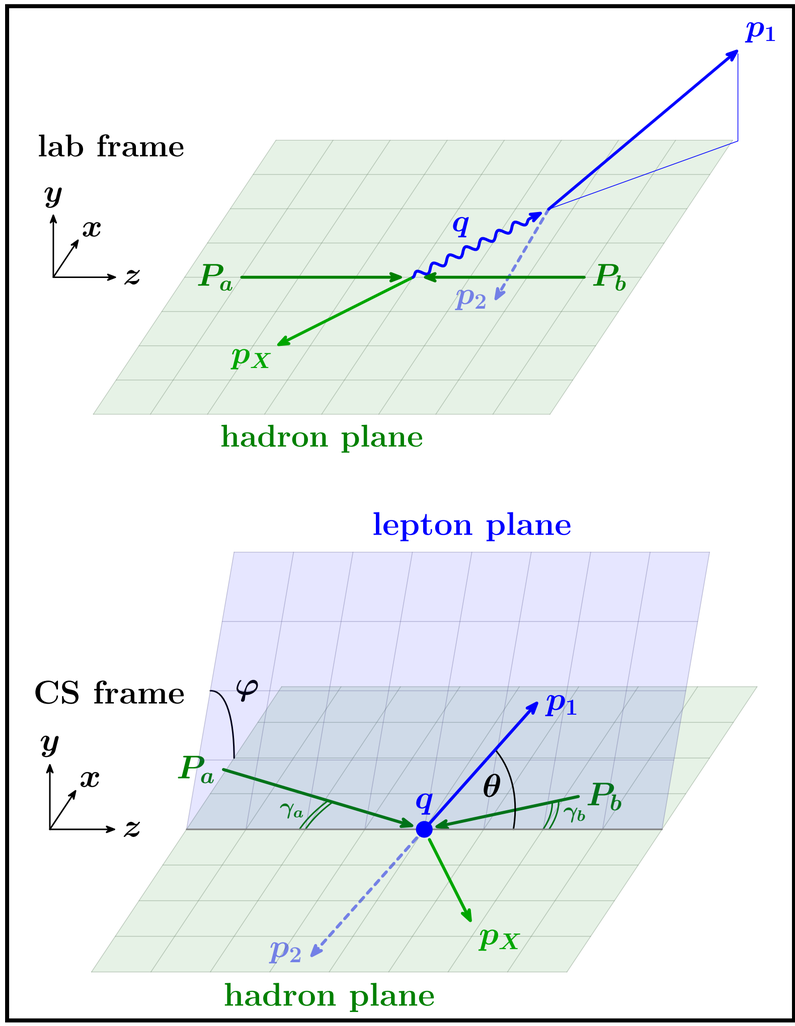
General dependence on the angular distributions of leptons in the final state

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^4q\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi} = \frac{3}{16\pi}\frac{\mathrm{d}\sigma_{\mathrm{incl}}}{\mathrm{d}^4q}\left[1+\cos^2\theta+\frac{A_0}{2}\left(1-3\cos^2\theta\right)+\sum_{i=1}^{7}A_i\,g_i(\theta,\varphi)\right]$$

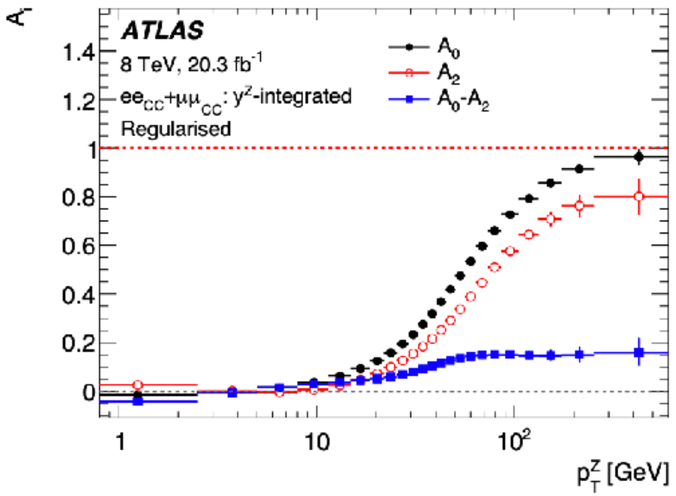
inclusive cross section
$\frac{\mathrm{d}\sigma_{\mathrm{incl}}}{\mathrm{d}^4q}(q, P_a, P_b)$

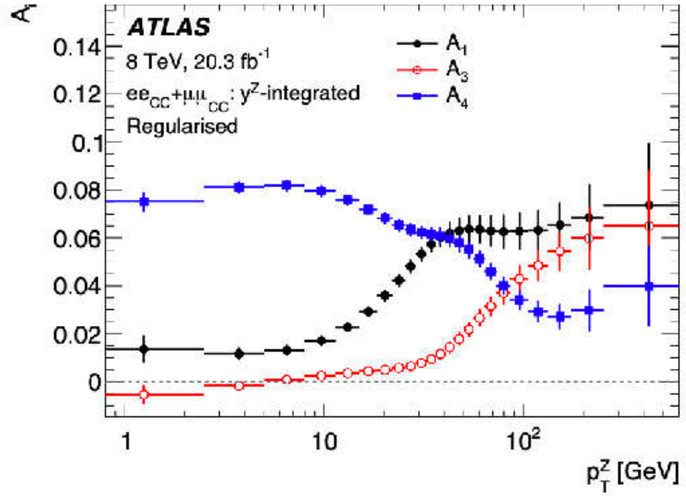
angular coefficients
$A_i(q^2, q_T, y_Z)$

spherical harmonics
$g_i(\theta, \varphi)$



[M.Ebert, J.Michel, I.Stewart, F.Tackmann '20]

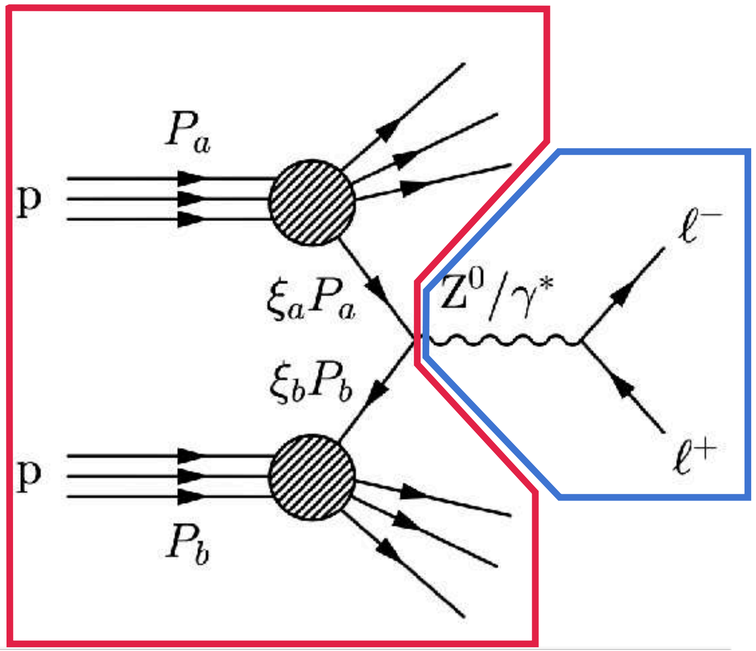




[Nikhef Thesis D.Ponomarenko '24]

Deriving angular coefficients

1. Differential cross section in terms of leptonic and hadronic tensors



$$\frac{\mathrm{d}\sigma(\Theta)}{\mathrm{d}^4 q\, \mathrm{d}\mathcal{O}} = \frac{1}{2E_{\mathrm{cm}}^2} L_{\mu\nu}(q, \mathcal{O}, \Theta) W^{\mu\nu}(q, P_a, P_b)$$

2. Project $W^{\mu\nu}$ onto helicity states of the V boson:

$$W_{\lambda\lambda'} = \epsilon_\lambda^\mu \epsilon_{\lambda'}^{*\nu} W_{\mu\nu}$$

Helicity density matrix

Helicity cross sections

Helicity density matrix $W_{\lambda\lambda'} = \epsilon^{\mu}_{\lambda}\epsilon^{*\nu}_{\lambda'}\, W_{\mu\nu}$ Hadronic tensor





Structure functions

$$
\begin{aligned}
W_{-1} &= W_{++} + W_{--} \\
W_0 &= 2W_{00} \\
W_1 &= -1/\sqrt{2}(W_{+0} + W_{0+} + W_{-0} + W_{0-}) \\
W_2 &= -2(W_{+-} + W_{-+}) \\
W_3 &= -\sqrt{2}(W_{+0} + W_{0+} - W_{-0} - W_{0-}) \\
W_4 &= 2(W_{++} - W_{--}) \\
W_5 &= -i(W_{+-} - W_{-+}) \\
W_6 &= -i/\sqrt{2}(W_{+0} - W_{0+} - W_{-0} + W_{0-}) \\
W_7 &= -\sqrt{2}i(W_{+0} - W_{0+} + W_{-0} - W_{0-})\,.
\end{aligned}
$$

Helicity cross sections:

$$\frac{\mathrm{d}\sigma_i}{\mathrm{d}^4 q} = \frac{1}{2E_{\mathrm{cm}}^2} L_{\pm(i)}(q^2, \theta, \phi) W_i(q, P_a, P_b)$$

Angular coefficients:

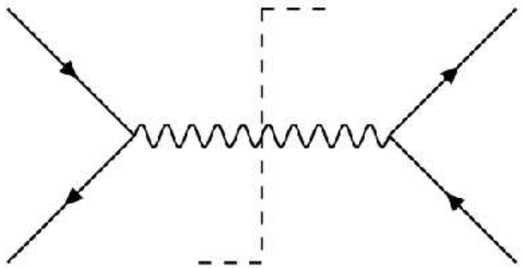
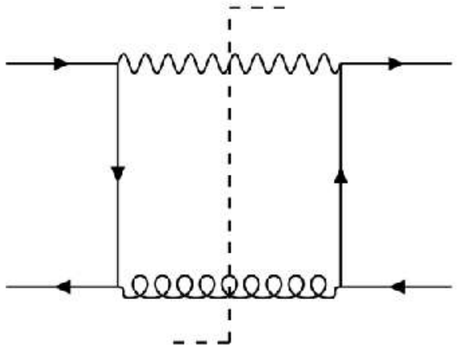
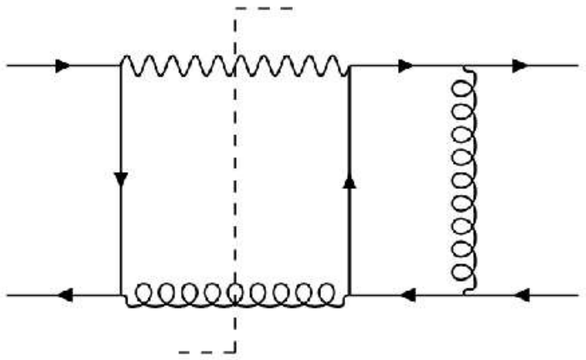
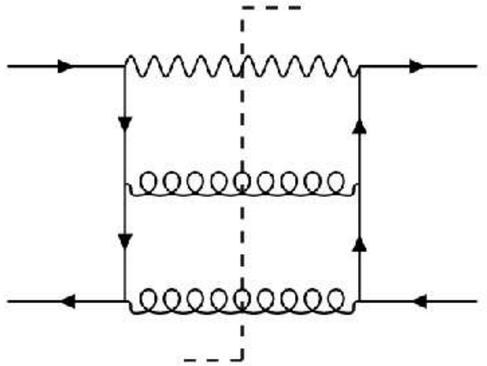
$$A_i = \frac{\mathrm{d}\sigma_i}{\mathrm{d}\sigma_{-1} + 1/2\,\mathrm{d}\sigma_0}$$

Angular distribution ↔ helicity structure ← QCD → **Goal: compute** $\dfrac{\mathrm{d}\sigma_i}{\mathrm{d}^4 q}$

Complete Analytic Calculation of Drell-Yan Angular Coefficients at $\mathcal{O}(\alpha_s^2)$

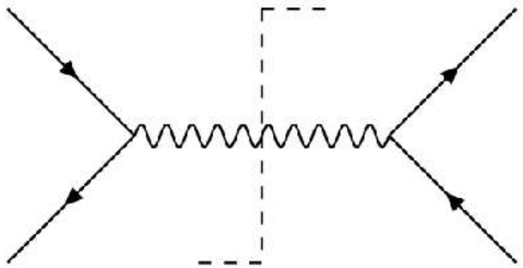
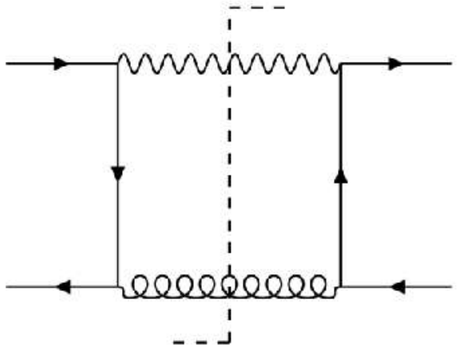
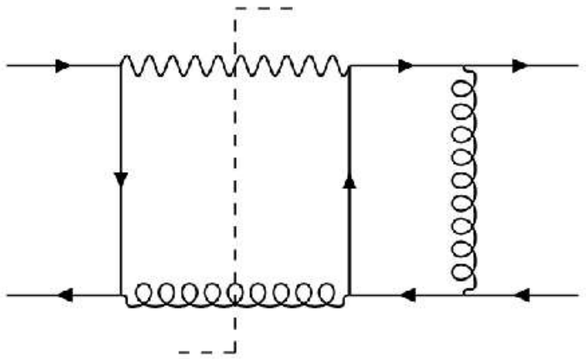
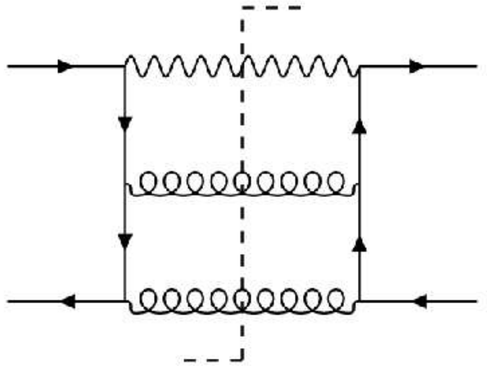
Radiative corrections

Perturbative expansion in strong coupling: $\frac{\mathrm{d}\sigma}{\mathrm{d}q_T} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n \frac{\mathrm{d}\sigma^{(n)}}{\mathrm{d}q_T}$

$\mathcal{O}(\alpha_s^0)$ $q_T = 0$	$\mathcal{O}(\alpha_s^1)$ $q_T \neq 0$	$\mathcal{O}(\alpha_s^2)$ $q_T \neq 0$	
			
LO= Born × Born	NLO = $\mathrm{LO}_1 \times \mathrm{LO}_1$	NNLO = $\mathrm{LO}_1 \times \mathrm{NLO}_1 + \mathrm{LO}_2 \times \mathrm{LO}_2$	
Born	R	RV	RR
$q\bar{q} \to V$	$q\bar{q} \to Vg, qg \to Vq$	$q\bar{q} \to Vg, qg \to Vq$	$q\bar{q} \to Vq\bar{q}, qq \to Vqq,$ $q\bar{q} \to Vgg, gg \to Vq\bar{q},$ $qg \to Vqg$

Radiative corrections

Perturbative expansion in strong coupling: $\dfrac{\mathrm{d}\sigma}{\mathrm{d}q_T} = \sum_{n=0}^{\infty} \left(\dfrac{\alpha_s}{2\pi}\right)^n \dfrac{\mathrm{d}\sigma^{(n)}}{\mathrm{d}q_T}$

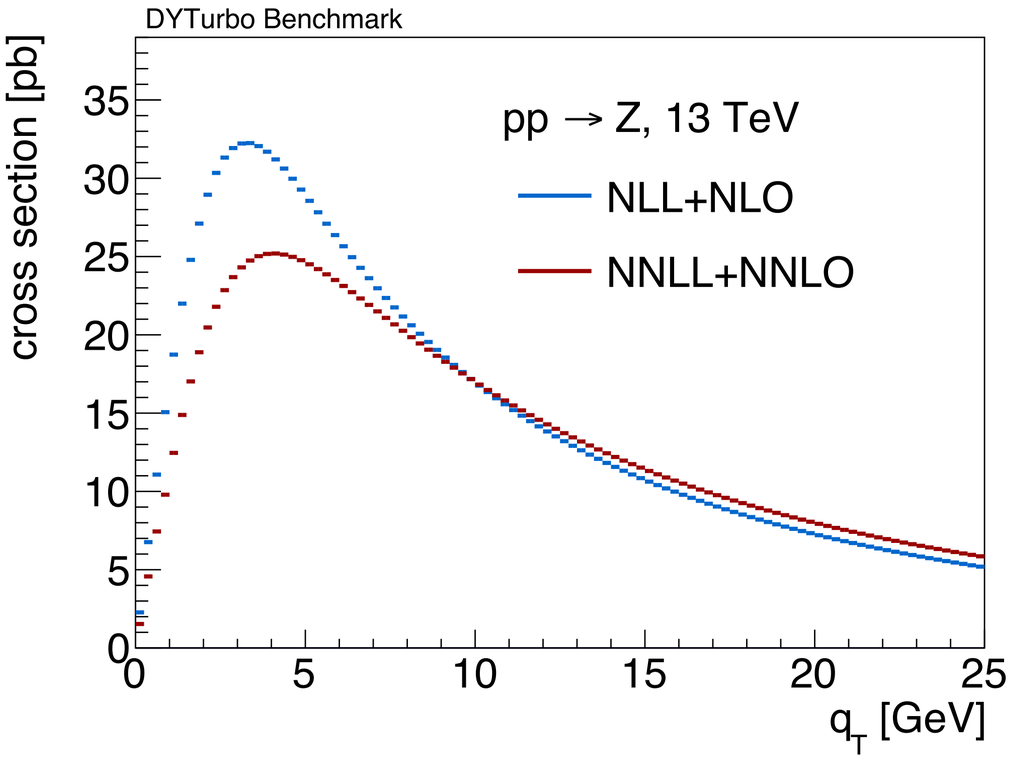
$\mathcal{O}(\alpha_s^0)$ $q_T = 0$	$\mathcal{O}(\alpha_s^1)$ $q_T \neq 0$	$\mathcal{O}(\alpha_s^2)$ $q_T \neq 0$	
			
LO= Born × Born	NLO = $\mathrm{LO}_1 \times \mathrm{LO}_1$	NNLO = $\mathrm{LO}_1 \times \mathrm{NLO}_1 + \mathrm{LO}_2 \times \mathrm{LO}_2$	
Born	R	RV	RR
$q\bar{q} \to V$	$q\bar{q} \to Vg,\ qg \to Vq$	$q\bar{q} \to Vg,\ qg \to Vq$	$q\bar{q} \to Vq\bar{q},\ qq \to Vqq,$ $q\bar{q} \to Vgg,\ gg \to Vq\bar{q},$ $qg \to Vqg$

Value of analytical expressions

Example of DYTurbo:

MC only: Full (integrated) incl. cross section at NNLO:
8 hr x 100 bins x 20 threads = 16 000 CPU-hours

Using partial analytic: 68 CPU-hours

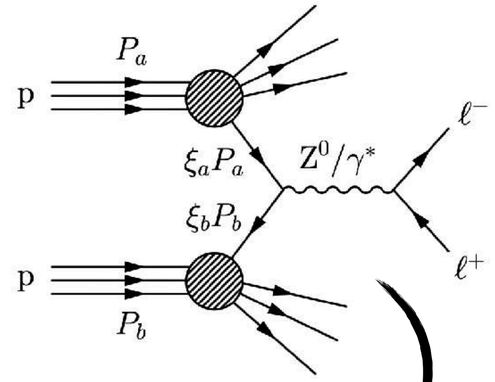


[S.Camarda, M.Boonekamp, G.Bozzi et al. '20]

Analytical

- ✦ Once and done!
- ✦ Faster evaluation (especially for 9 $\sigma_{i,ab}$!)
- ✦ Reusable and modifiable
- ✦ Physical insight

Only formulas



Complete Analytic Calculation of Drell-Yan Angular Coefficients at $\mathcal{O}(\alpha_s^2)$

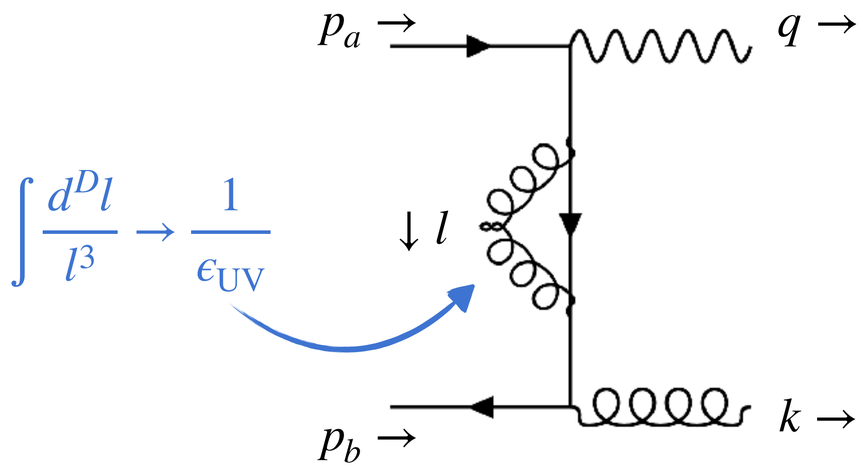
$$\frac{\mathrm{d}\sigma}{\mathrm{d}^4q\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi} \sim \left[1+\cos^2\theta+\frac{A_0}{2}\left(1-3\cos^2\theta\right)+\sum_{i=1}^{7}A_i\,g_i(\theta,\varphi)\right]$$

$$\sigma_i \sim \alpha_s^2$$

Pole structure (dimreg)

Don't worry, we didn't forget about them

RV	RR	α_s counter term	PDF counter term
$\frac{1}{\epsilon_{\text{UV}}} + \frac{a}{\epsilon_{\text{IR}}} + \frac{1}{\epsilon_{\text{IR}}^2}$	$\frac{b}{\epsilon_{\text{IR}}} - \frac{1}{\epsilon_{\text{IR}}^2}$	$-\frac{1}{\epsilon_{\text{UV}}}$	$-\frac{a+b}{\epsilon_{\text{IR}}}$



Pole structure (dimreg)

Renormalization, cancel UV

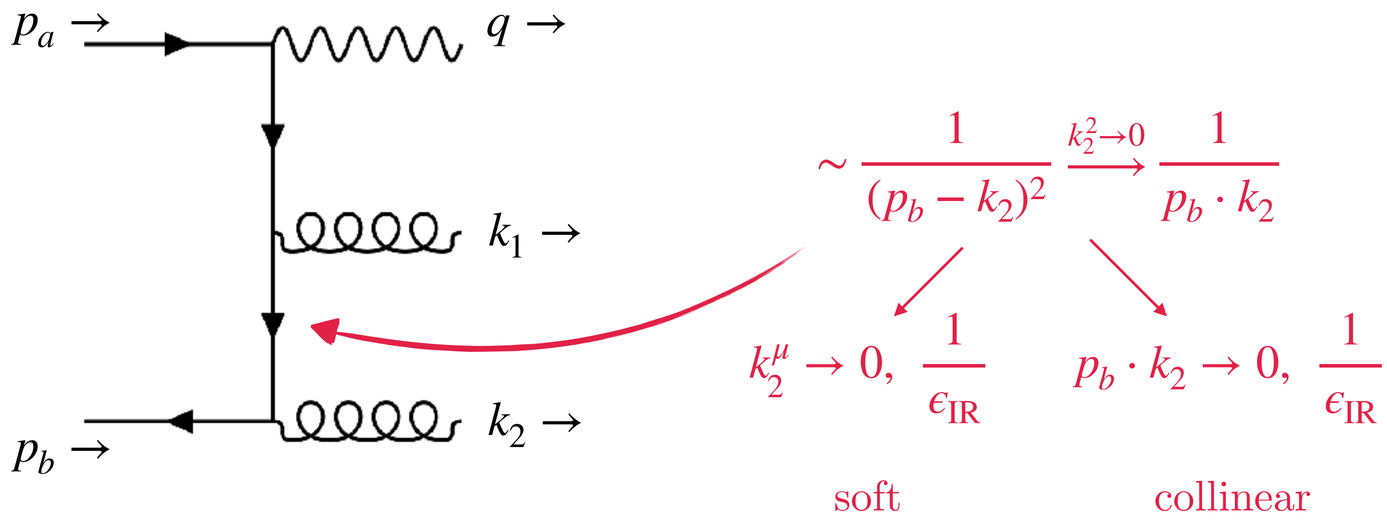
RV	RR	α_s counter term	PDF counter term
$\frac{1}{\epsilon_{\rm UV}} + \frac{a}{\epsilon_{\rm IR}} + \frac{1}{\epsilon_{\rm IR}^2}$	$\frac{b}{\epsilon_{\rm IR}} - \frac{1}{\epsilon_{\rm IR}^2}$	$-\frac{1}{\epsilon_{\rm UV}}$	$-\frac{a+b}{\epsilon_{\rm IR}}$

$$\alpha_s^{\rm bare} \left[\quad \right] = \left(\alpha_s(\mu_R) - \frac{\alpha_s^2(\mu_R)}{\epsilon} \right) \left[\quad \right]$$

Pole structure (dimreg)

Don't worry, we didn't forget about them

RV	RR	α_s counter term	PDF counter term
$\frac{1}{\epsilon_{\rm UV}} + \frac{a}{\epsilon_{\rm IR}} + \frac{1}{\epsilon_{\rm IR}^2}$	$\frac{b}{\epsilon_{\rm IR}} - \frac{1}{\epsilon_{\rm IR}^2}$	$-\frac{1}{\epsilon_{\rm UV}}$	$-\frac{a+b}{\epsilon_{\rm IR}}$



Pole structure (dimreg)

Same thing, but for IR poles

RV	RR	α_s counter term	PDF counter term
$\frac{1}{\epsilon_{\rm UV}} + \frac{a}{\epsilon_{\rm IR}} + \frac{1}{\epsilon_{\rm IR}^2}$	$\frac{b}{\epsilon_{\rm IR}} - \frac{1}{\epsilon_{\rm IR}^2}$	$-\frac{1}{\epsilon_{\rm UV}}$	$-\frac{a+b}{\epsilon_{\rm IR}}$

$$f_k^{[h],\text{bare}}(\xi)\ \alpha_s(\mu_R) \otimes \Big[\ \Big] = \left(f_k^{[h]}(\xi,\mu_F^2)\ \alpha_s(\mu_R) + \alpha_s^2(\mu_R)\frac{1}{\epsilon} f(\mu_F^2) \otimes P^{(0)} \right) \otimes \Big[\ \Big]$$

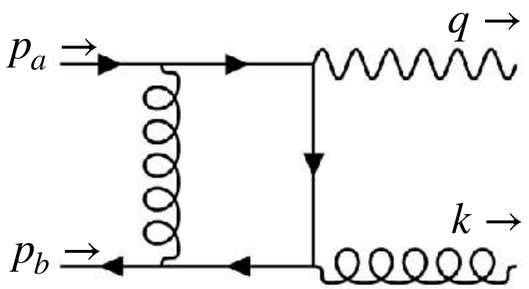
Canceling all poles in RV and RR → tedious, is there a better way?

V+jet threshold factorization as an analytic subtraction

A better way to manage poles

Separate the partonic hel. cross section into singular and non-singular terms

RV



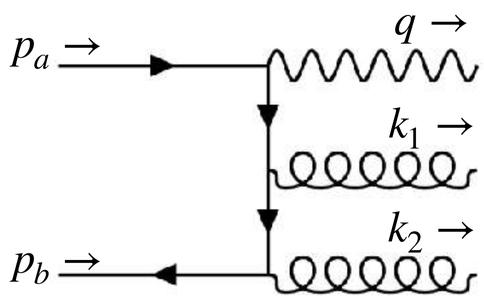
$$\hat{\sigma}_{i,ab} = \hat{\sigma}^{\text{sing}}_{i,ab} + \underbrace{[\hat{\sigma}_{i,ab} - \hat{\sigma}^{\text{sing}}_{i,ab}]_{k^2>0}}_{\hat{\sigma}^{\text{non-sing}}_{i,ab}}$$

$$\sim \delta(k^2), \left[\frac{\ln^n(k^2/\nu^2)}{k^2}\right]_+ \qquad \sim \mathcal{O}(1)$$

- All contributions from RV,
- Part of RR,
- α_s CT,
- Part of PDF CT

- Rest of RR,
- Rest of PDF CT

RR



$$k^\mu = k_1^\mu + k_2^\mu$$

- ✦ Poles cancel indep. for each term
- ✦ Easier to convolve with PDFs and integrate numerically

V+jet threshold factorization as an analytic subtraction

Singular terms using SCET factorization formula:

$$\hat{\sigma}^{\text{sing}}_{i,ab} \propto H_{i,ab}\, J_{ab} \otimes S_{ab}$$

SCET factorization formula is fully finite in ϵ

$$\downarrow$$

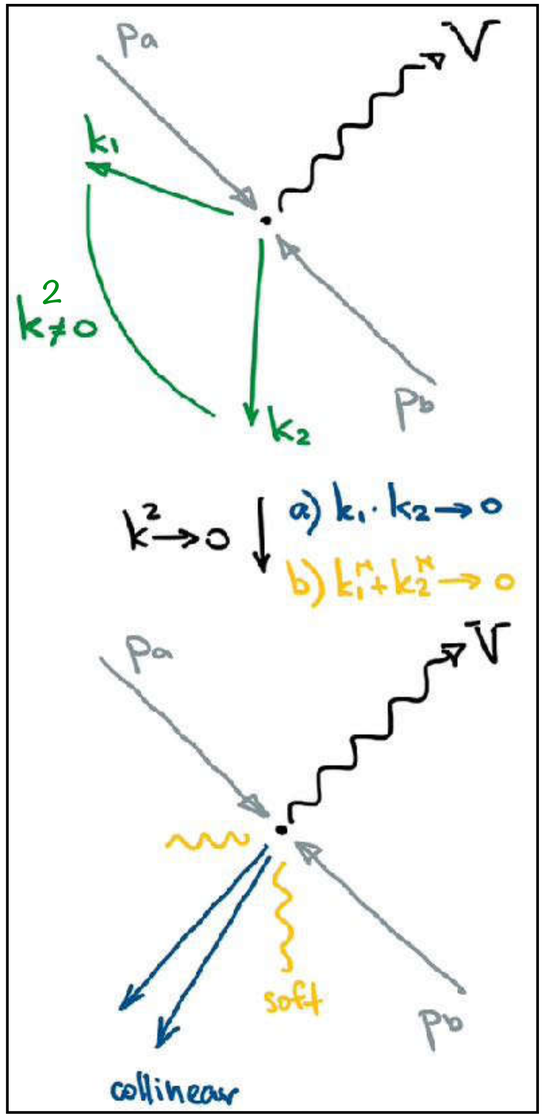
$$\hat{\sigma}^{\text{sing},(2)}_{i,ab} \propto H^{(2)}_{i,ab} + H^{(1)}_{i,ab}\,(J^{(1)}_{ab} + S^{(1)}_{ab})$$

[T.Becher, C.Lorentzen, M.Schwartz '13]

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

finite ? finite ✓ finite ✓ finite ✓

Goal 1: compute $H^{(2)}_{i,ab}$ from finite RV amplitudes



V+jet threshold factorization as an analytic subtraction

Non-singular terms, since we are only left terms with RR:

$$\hat{\sigma}^{\text{non-sing},(2)}_{i,ab} = \left[\hat{\sigma}^{\text{full}}_{i,ab}\right]_{\text{RR}} - \hat{\sigma}^{\text{sing},(2)}_{i,ab}\Big|_{k^2>0}$$

Goal 2: compute $\left[\hat{\sigma}^{\text{full}}_{i,ab}\right]_{\text{RR}}$

Now onto RV!



Spinor-helicity formalism 101

Simple action ⟶ **Complicated** and **redundant** Feynman rules ⟶ **Simple** amplitude squares

$$S = -\frac{1}{4}\int \mathrm{d}^4x F^a_{\mu\nu} F^{a\,\mu\nu} \qquad \mathscr{A} = -ig_s^2\{\text{horrible mess}\} \qquad |\overline{\mathscr{A}}|^2 = \frac{9g_s^4}{2}\left(3 - \frac{su}{t^2} - \frac{ut}{s^2} - \frac{st}{u^2}\right)$$

[T.Bader '21]

Exchanging the four-component spinor $u(p)$ for two bispinors

$$u_+(p) = |p\rangle = \frac{1+\gamma_5}{2}u(p) \text{ — positive-helicity spinor}$$

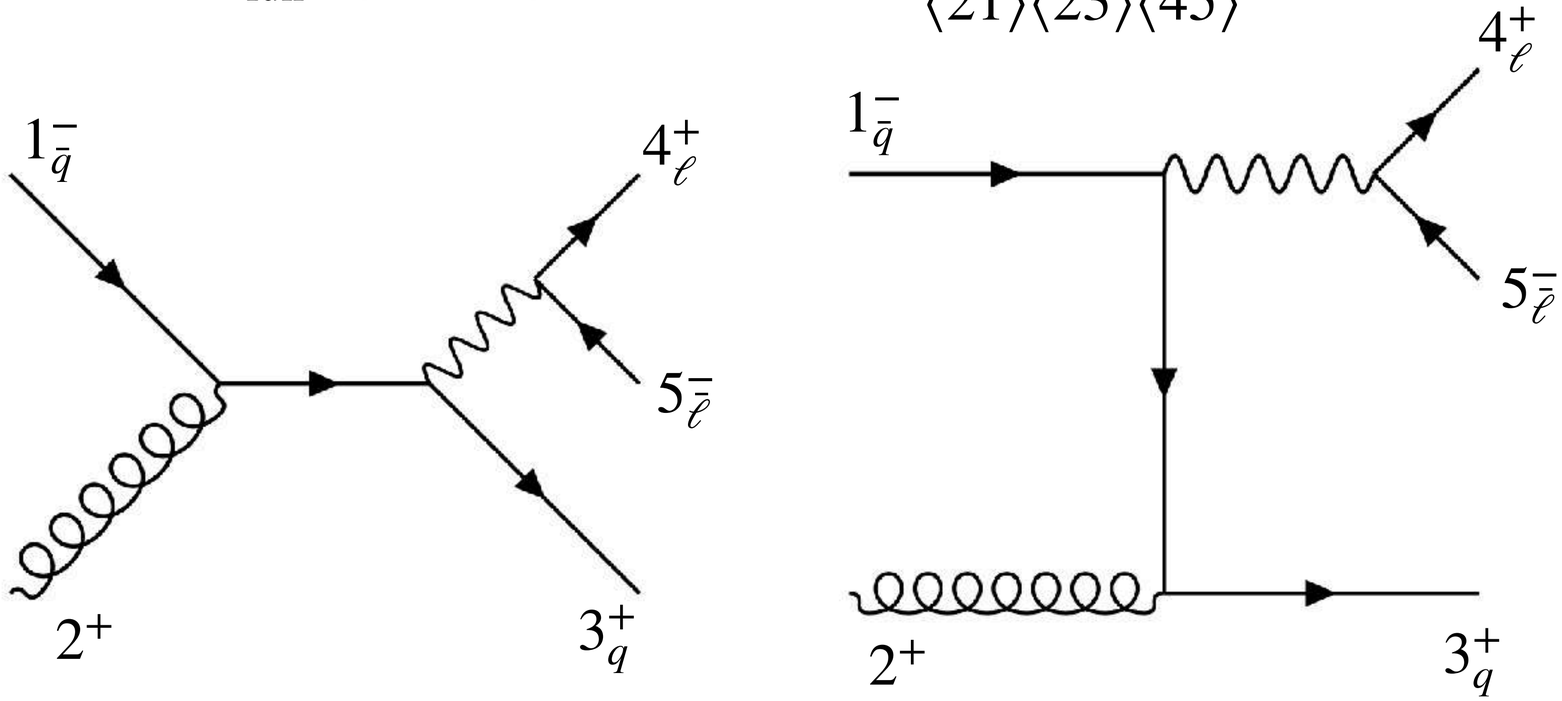
$$u_-(p) = |p] = \frac{1-\gamma_5}{2}u(p) \text{ — negative-helicity spinor}$$

Result: helicity-dependent, simple and compact amplitude $\mathscr{A}(1^{\pm},2^{\pm},\ldots)$

Spinor-helicity formalism 101

$$\mathscr{A}^{(1)}_{\text{full}}(1^-_{\bar{q}};2^+;3^+_q;4^+_\ell,5^-_{\bar{\ell}}) = 2\sqrt{2}\frac{\langle 15\rangle^2}{\langle 21\rangle\langle 23\rangle\langle 45\rangle}$$

[I.Moult, I.Stewart, F.Tackmann, W.Waalewijn '15]



✓ Only D=4 → SCET fact. formula finite

✗ Only massless external states → see next slide

Partonic $\mathscr{A}$ using spinor-helicity formalism

Our main idea

Factorize the spinor-helicity amplitude $p_1 p_2 \to p_3 V(q \to p_4 p_5)$

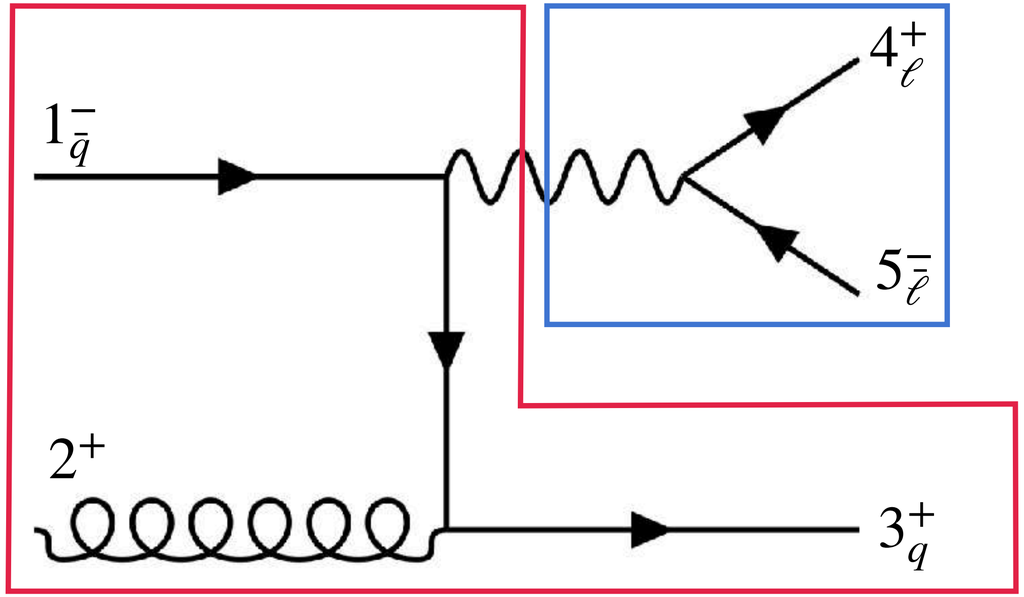
$$\mathscr{A}_{\text{full}}(1^{\lambda_g}_{g}, 2^{\lambda_q}_{\bar{q}}, 3^{-\lambda_q}_{q}, 4^{\lambda_\ell}_{\ell}, 5^{-\lambda_\ell}_{\bar{\ell}}) = \sum_{\lambda} \mathscr{A}_{\lambda}(1^{\lambda_g}_{g}, 2^{\lambda_q}_{\bar{q}}, 3^{-\lambda_q}_{q}) \, \mathscr{A}_{\ell\lambda}(4^{\lambda_\ell}_{\ell}, 5^{-\lambda_\ell}_{\bar{\ell}})$$

Z helicity amplitudes:

$$\mathscr{A}_{\lambda}(1^{\lambda_g}_{g}, 2^{\lambda_q}_{\bar{q}}, 3^{-\lambda_q}_{q}) = \{\mathscr{A}_0, \mathscr{A}_+, \mathscr{A}_-\}$$

Projected leptonic current:

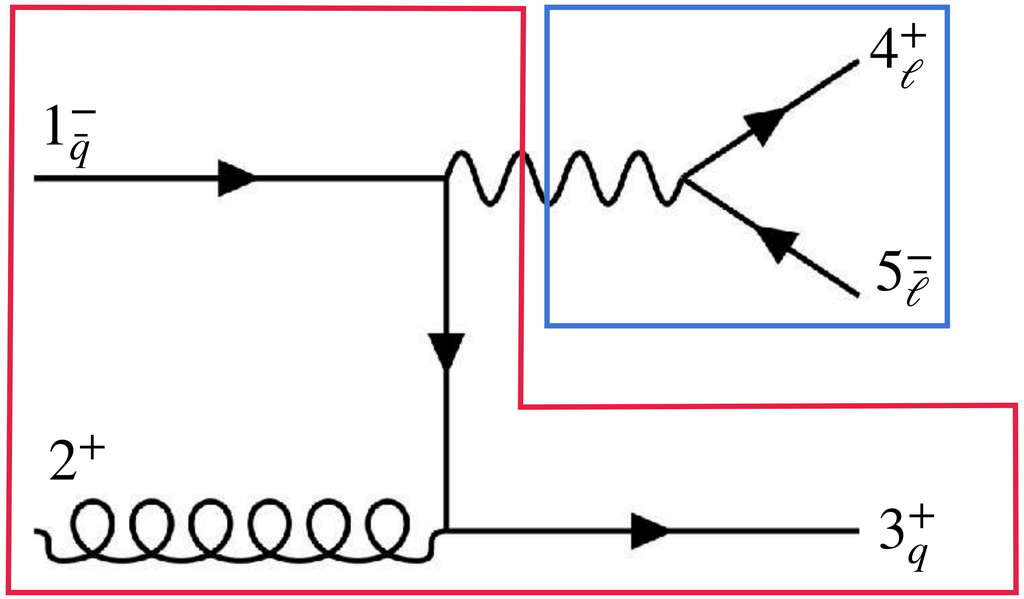
$$\mathscr{A}_{\ell\lambda}(4^{\lambda_\ell}_{\ell}, 5^{-\lambda_\ell}_{\bar{\ell}}) = \epsilon^{\mu}_{\lambda} \, L_{\lambda_\ell,\mu}(4^{\lambda_\ell}_{\ell}, 5^{-\lambda_\ell}_{\bar{\ell}})$$



Partonic $\mathscr{A}$ using spinor-helicity formalism

Our main idea

$$\mathscr{A}^{(1)}_{\text{full}}(1^-_{\bar{q}};2^+;3^+_q;4^+_\ell,5^-_{\bar{\ell}}) = 2\sqrt{2}\frac{\langle 15\rangle^2}{\langle 21\rangle\langle 23\rangle\langle 45\rangle}$$



$$\sim \begin{array}{|r|l|} \hline \mathscr{A}^{(1)}_0(1^-_{\bar{q}};2^+;3^+_q) & \mathscr{A}_{\ell,0}(4^+_\ell,5^-_{\bar{\ell}}) \\ +\mathscr{A}^{(1)}_+(1^-_{\bar{q}};2^+;3^+_q) & \mathscr{A}_{\ell,+}(4^+_\ell,5^-_{\bar{\ell}}) \\ +\mathscr{A}^{(1)}_-(1^-_{\bar{q}};2^+;3^+_q) & \mathscr{A}_{\ell,-}(4^+_\ell,5^-_{\bar{\ell}}) \\ \hline \end{array}$$

- ✦ Have to solve for
- ✦ Universal

Example sol. for LO_1:

$$\mathscr{A}^{(1)}_0 = -\left[g_Z(a_q - v_q)\right]\frac{g_S}{\sqrt{2}}\frac{s(Q^2+s)}{\sqrt{s(Q^2-u)(Q^2-t)}}\frac{\langle 23\rangle}{\langle 12\rangle^2}$$

$$\mathscr{A}^{(1)}_\pm = \mp\left[g_Z(a_q - v_q)\right]\frac{g_S}{2}\frac{\sqrt{s(Q^2-u)(Q^2-t)}(t+u) \mp Qs(t-u)}{\sqrt{tu(Q^2-u)(Q^2-t)}}\frac{\langle 23\rangle}{\langle 12\rangle^2}$$

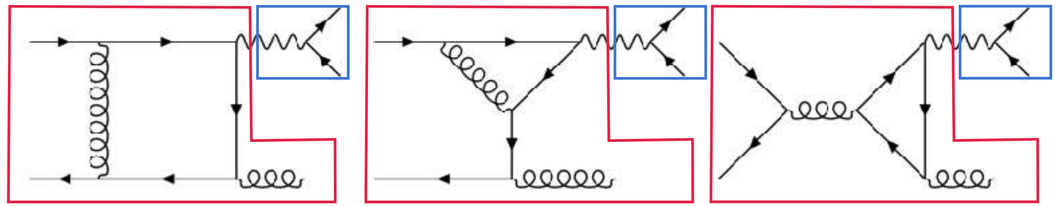
Partonic $\mathscr{A}$ using spinor-helicity formalism

Our main idea

✦ Works at any order in α_s:

$$
\begin{aligned}
A^{(2)}_{\text{full,fin}}(1^+;2^+_q,3^-_{\bar q};4^+_\ell,5^-_{\bar\ell}) &= A^{(1)}_{\text{full}}(1^+,2^+_{\bar q},3^-_q;4^+_\ell,5^-_{\bar\ell}) \\
&\times\left\{\frac{C_A}{2}\left(-L_{12}^2-L_{13}^2+3L_{13}-7+\frac{\pi^2}{3}\right)+\left(C_F-\frac{C_A}{2}\right)\left(-L_{23}^2+3L_{45}-8+\frac{\pi^2}{6}\right)\right. \\
&+C_A\left[-\mathrm{Ls}_{-1}\left(\frac{s_{12}}{s_{45}},\frac{s_{13}}{s_{45}}\right)+\frac{\langle3|24|5\rangle}{\langle35\rangle s_{45}}\mathrm{L}_0\left(\frac{s_{13}}{s_{45}}\right)-\frac{1}{2}\frac{\langle3|24|5\rangle^2}{\langle35\rangle^2 s_{45}^2}\mathrm{L}_1\left(\frac{s_{13}}{s_{45}}\right)\right] \\
&+(C_A-2C_F)\left[\mathrm{Ls}_{-1}\left(\frac{s_{12}}{s_{45}},\frac{s_{23}}{s_{45}}\right)+\left(\frac{\langle25\rangle^2\langle13\rangle^2}{\langle12\rangle^2\langle35\rangle^2}-\frac{\langle15\rangle\langle23\rangle+\langle13\rangle\langle25\rangle}{\langle12\rangle\langle35\rangle}\right)\mathrm{Ls}_{-1}\left(\frac{s_{13}}{s_{45}},\frac{s_{23}}{s_{45}}\right)\right. \\
&+\frac{2[12]\langle25\rangle\langle13\rangle}{\langle35\rangle s_{45}}\mathrm{L}_0\left(\frac{s_{13}}{s_{45}}\right)+\frac{\langle25\rangle^2[2|1|3\rangle\langle13\rangle}{\langle12\rangle\langle35\rangle^2 s_{13}}\mathrm{L}_0\left(\frac{s_{45}}{s_{13}}\right)-\frac{\langle3|21|5\rangle\langle25\rangle\langle13\rangle}{\langle12\rangle\langle35\rangle^2 s_{23}}\mathrm{L}_0\left(\frac{s_{45}}{s_{23}}\right) \\
&-\frac{\langle13\rangle^2[1|2|5\rangle^2}{2\langle35\rangle^2 s_{13}^2}\mathrm{L}_1\left(\frac{s_{45}}{s_{13}}\right)-\frac{[1|2|3\rangle[4|1|5\rangle\langle45\rangle\langle13\rangle}{\langle35\rangle^2 s_{23}^2}\mathrm{L}_1\left(\frac{s_{45}}{s_{23}}\right) \\
&\left.\left.+\frac{[14](\![12][34]+[13][24])\langle13\rangle\langle45\rangle}{2\langle35\rangle^2[13][23][45]}\right]\right\},
\end{aligned}
$$

↕



$+\ldots$

[I.Moult, I.Stewart, F.Tackmann, W.Waalewijn '15]

$$
\sim \begin{array}{|l|l|}
\hline
\mathscr{A}_0^{(2)}(1^-_{\bar q};2^+;3^+_q) & \mathscr{A}_{\ell,0}(4^+_\ell,5^-_{\bar\ell}) \\
+\mathscr{A}_+^{(2)}(1^-_{\bar q};2^+;3^+_q) & \mathscr{A}_{\ell,+}(4^+_\ell,5^-_{\bar\ell}) \\
+\mathscr{A}_-^{(2)}(1^-_{\bar q};2^+;3^+_q) & \mathscr{A}_{\ell,-}(4^+_\ell,5^-_{\bar\ell}) \\
\hline
\end{array}
$$

✦ Have to solve for

✦ Same as for LO_1

Goal: $H^{(2)}_{i,ab} \sim W_{\lambda\lambda'} = \sum_{(\ldots)} \mathscr{A}^{(1)}_{\lambda}\mathscr{A}^{(2)*}_{\lambda'} + \mathscr{A}^{(2)}_{\lambda}\mathscr{A}^{(1)*}_{\lambda'}$

helicity density matrix

summed over all helicities

✦ **Get** $H^{(2)}_{i,ab}$ ✓

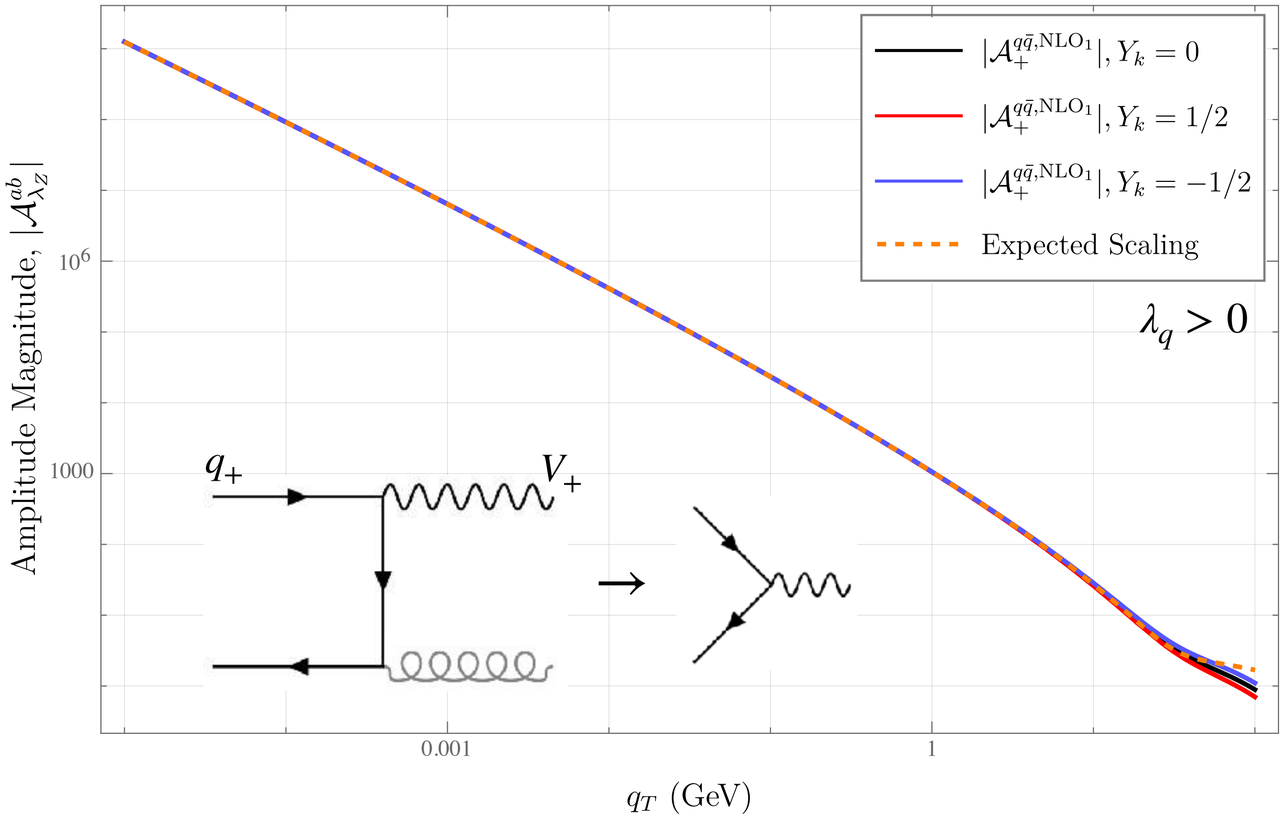
Power counting for NLO_1 Z-helicity amplitudes

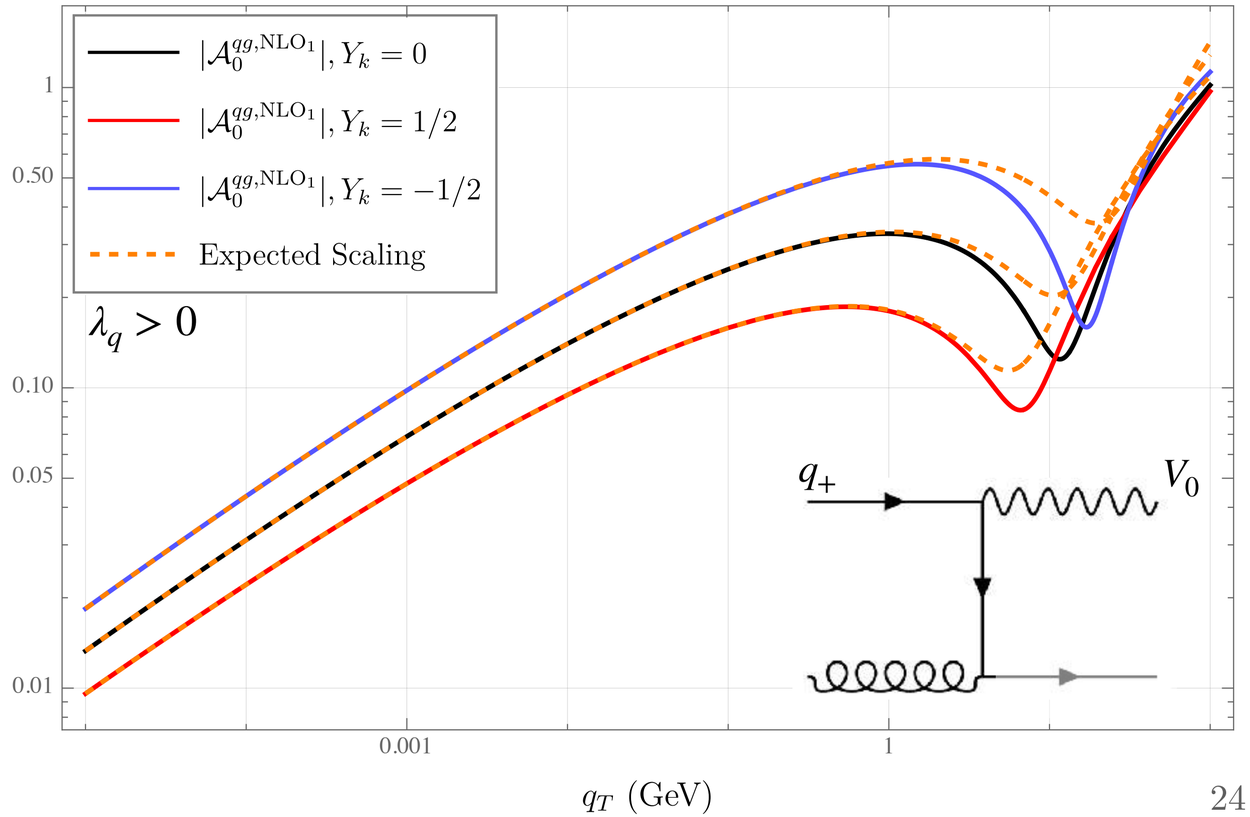
Scaling (soft) in the low-q_T/Q limit

$$\lambda = \frac{q_T}{Q} \ll 1$$

$$|\mathscr{A}_+|_{q\bar{q}\to Vg} \sim \mathcal{O}\left(\lambda^{-1}\left(1+\log(\lambda)+\log(\lambda)^2\right)\right),$$

$$|\mathscr{A}_0|_{qg\to Vq} \sim \mathcal{O}\left(\lambda^{1/2}\left(1+\log(\lambda)+\log(\lambda)^2\right)\right)$$





Recap and a breather

Singular term:

$$\hat{\sigma}^{\text{sing},(2)}_{i,ab} \propto H^{(2)}_{i,ab} + H^{(1)}_{i,ab}\,(J^{(1)}_{ab}+S^{(1)}_{ab})$$

↑

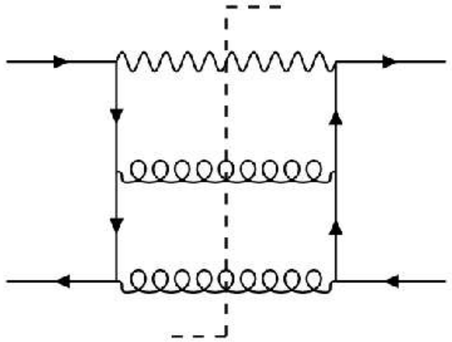
from massless spinor-helicity amplitudes

Non-singular term:

$$\hat{\sigma}^{\text{non-sing},(2)}_{i,ab} = \left[\hat{\sigma}^{\text{full}}_{i,ab}\right]_{\text{RR}} - \left.\hat{\sigma}^{\text{sing},(2)}_{i,ab}\right|_{k^2>0}$$

Goal 2: compute $\left[\hat{\sigma}^{\text{full}}_{i,ab}\right]_{\text{RR}}$ Now onto RR

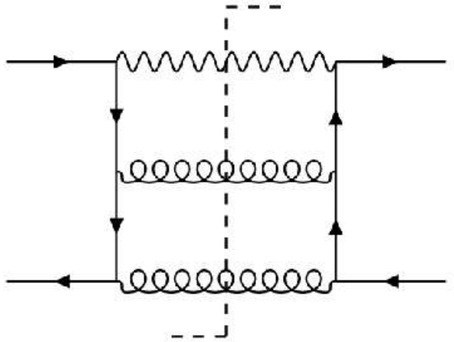




Main pipeline for RR

1. Evaluate RR contribution to W_i using conventional Feynman rules;

use spinor-helicity method as a check ✔ *



Main pipeline for RR

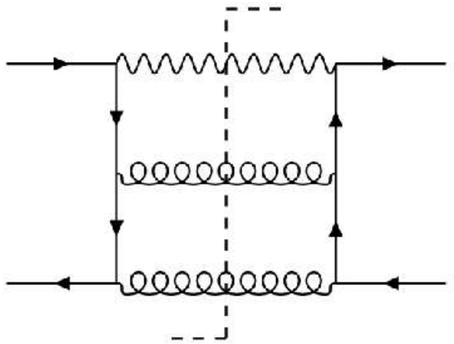
1. Evaluate RR contribution to W_i using conventional Feynman rules;

use spinor-helicity method as a check ✓ *

2. Integrate over hadronic radiation → **derive set of 6 unique master integrals**;

$$\int \frac{\mathrm{d}\phi_{(k\to k_1 k_2)}}{[(p_a-k_1)^2]^{j_1}[(p_b-k_2)^2]^{j_2}} \sim \int \frac{\mathrm{d}\Omega^{(3-2\epsilon)}}{(1-\beta_1\cos\theta_2\sin\theta_1-\beta_2\cos\theta_1)^{j_1}(1-\beta_3\cos\theta_2\sin\theta_1-\beta_4\cos\theta_1)^{j_2}}$$

[G.Somogyi '11]



Main pipeline for RR

1. Evaluate RR contribution to W_i using conventional Feynman rules;

use spinor-helicity method as a check ✓ *

2. Integrate over hadronic radiation → **derive set of 6 unique master integrals**;

3. Add PDF counter terms.

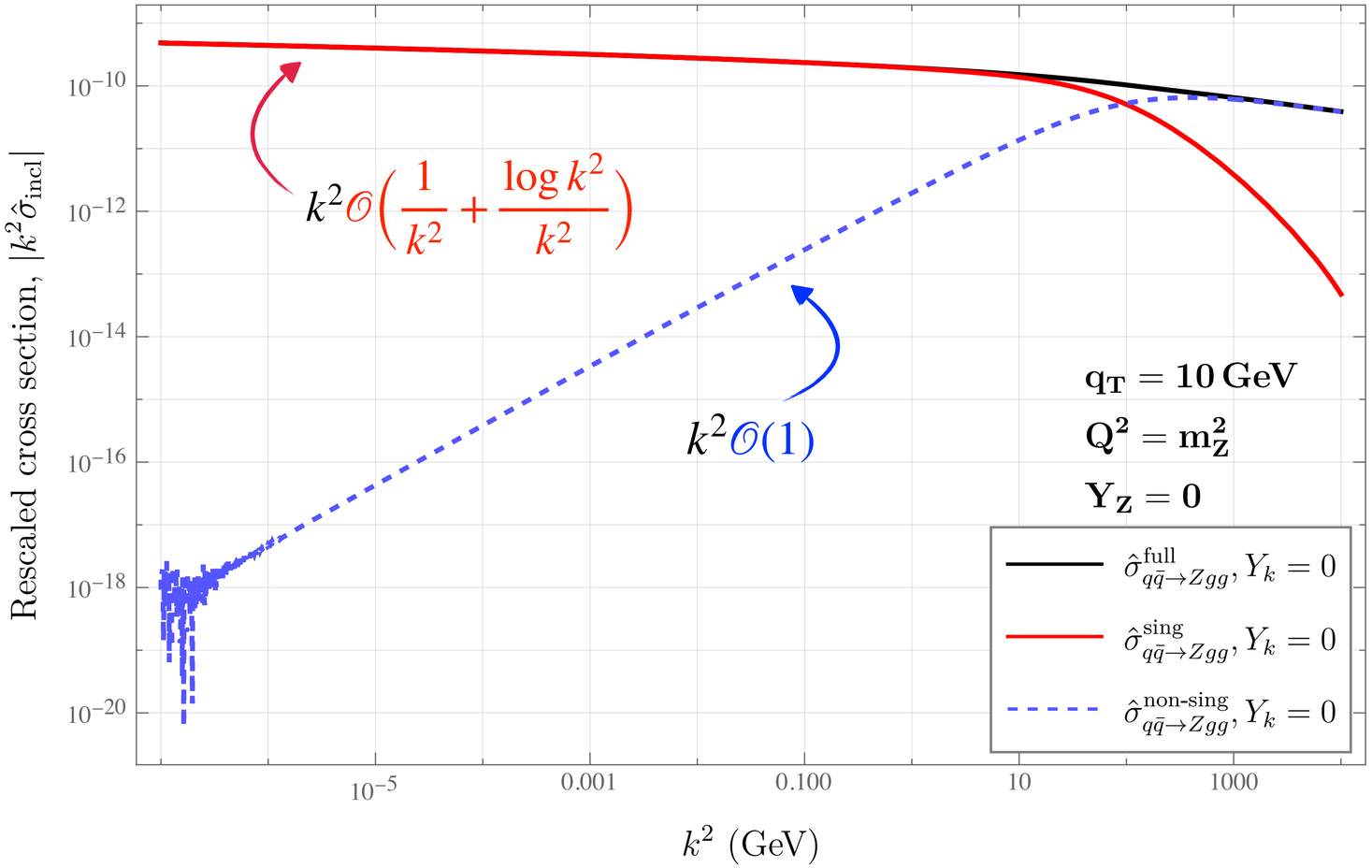
check if finite ✓

Important detail: γ_5 scheme for RR in $D = 4 - 2\epsilon$ dimensions

- ✦ non-anticommuting γ_5
- ✦ $\gamma_\mu\gamma_5 = \frac{i}{6}\epsilon_{\mu\nu\rho\sigma}\gamma^\nu\gamma^\rho\gamma^\sigma$ [Larin '93]

Non-singular cross section

Check k^2 scaling



$$\hat{\sigma}^{\rm non-sing,\,(2)}_{i,ab} = \left[\hat{\sigma}^{\rm full}_{i,ab}\right]_{\rm RR} - \hat{\sigma}^{\rm sing,\,(2)}_{i,ab}\Big|_{k^2>0}$$

$$\uparrow \qquad\qquad \uparrow$$

$$\mathcal{O}(1) \qquad\qquad \mathcal{O}\left(\frac{1}{k^2}+\frac{\log k^2}{k^2}\right)$$

✦ **Get** $\left[\hat{\sigma}^{\rm full}_{i,ab}\right]_{\rm RR}$ **as well as** $\hat{\sigma}^{\rm non-sing,\,(2)}_{i,ab}$ ✓

Final cross section

Results

$\left[\hat{\sigma}^{\text{full},(2)}_{i,ab}\right]_{\text{RR}}$ ✓

From scratch using conventional Feynman
+ master integrals
+ PDF renorm.

$H^{(2)}_{i,ab}$ ✓

From massless spinor-helicity amplitudes

$$\hat{\sigma}^{\text{non-sing},(2)}_{i,ab} = \left[\hat{\sigma}^{\text{full},(2)}_{i,ab}\right]_{\text{RR}} - \hat{\sigma}^{\text{sing},(2)}_{i,ab}\Big|_{k^2>0}$$

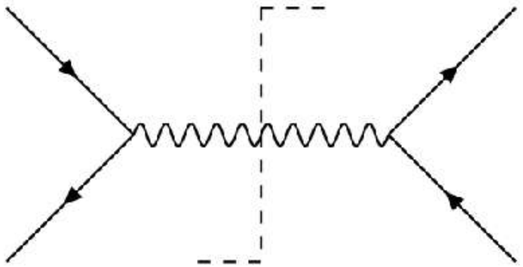
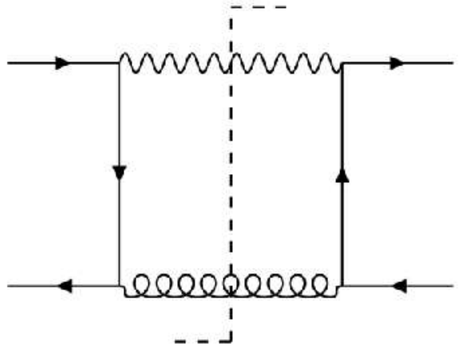
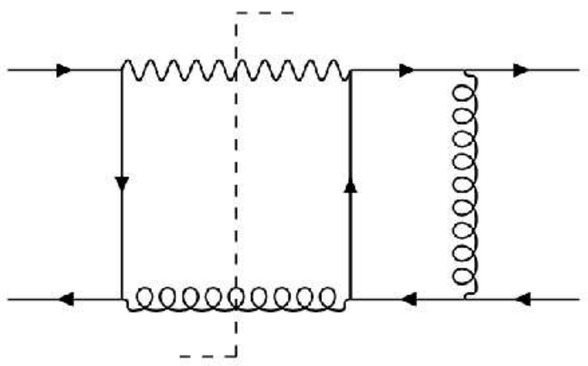
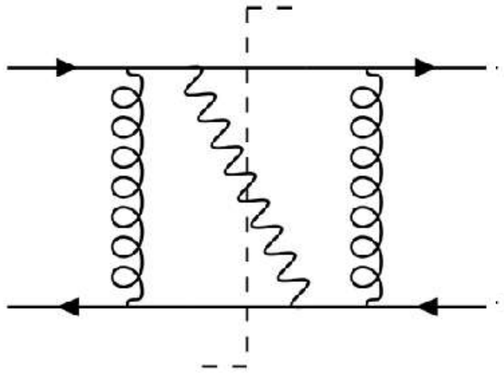
$$\hat{\sigma}^{\text{sing},(2)}_{i,ab} \propto H^{(2)}_{i,ab} + H^{(1)}_{i,ab}\left(J^{(1)}_{ab} + S^{(1)}_{ab}\right)$$

Assemble analytically using V+jet factorization:

$$\hat{\sigma}^{(2)}_{i,ab} = \hat{\sigma}^{\text{sing},(2)}_{i,ab} + \hat{\sigma}^{\text{non-sing},(2)}_{i,ab}$$



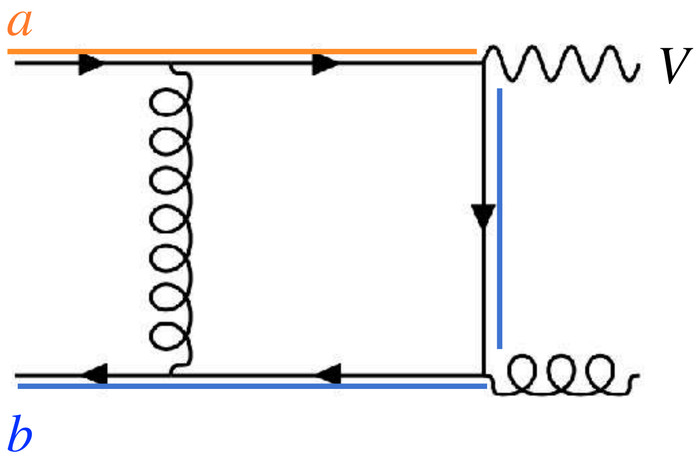
Remember this slide?

$\mathcal{O}(\alpha_s^0)$ $q_T = 0$	$\mathcal{O}(\alpha_s^1)$ $q_T \neq 0$	$\mathcal{O}(\alpha_s^2)$ $q_T \neq 0$	
		 	
LO= Born × Born	NLO = $LO_1 \times LO_1$	NNLO = $LO_1 \times NLO_1 + LO_2 \times LO_2$	
Born	R	RV	RR
$q\bar{q} \to V$	$q\bar{q} \to Vg, qg \to Vq$	$q\bar{q} \to Vg, qg \to Vq$	$q\bar{q} \to Vq\bar{q}, qq \to Vqq,$ $q\bar{q} \to Vgg, gg \to Vq\bar{q},$ $qg \to Vqg$

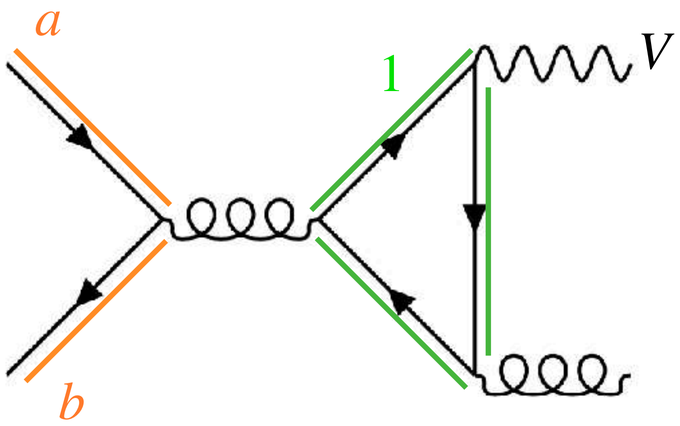
We calculated everything for $q = u$ and $VV' = ZZ$, what about different quarks and vector bosons?

Possible flavor couplings

Amplitudes and their EW couplings

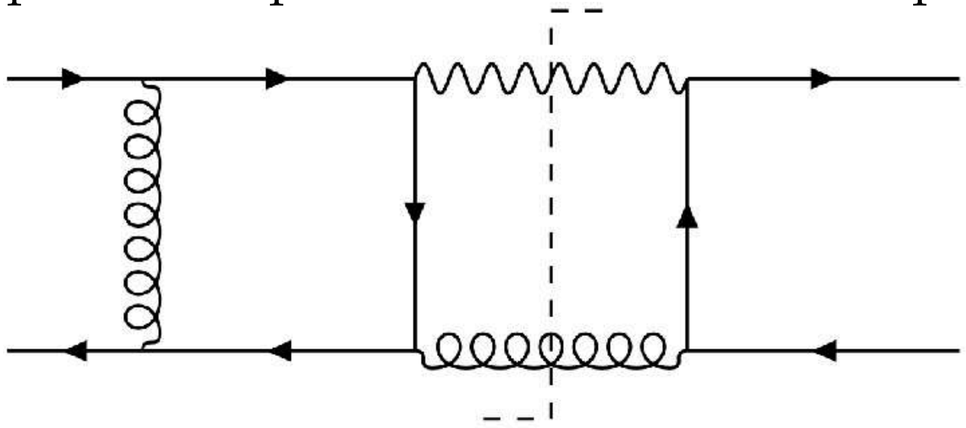


$$c_{f_a f_b}$$

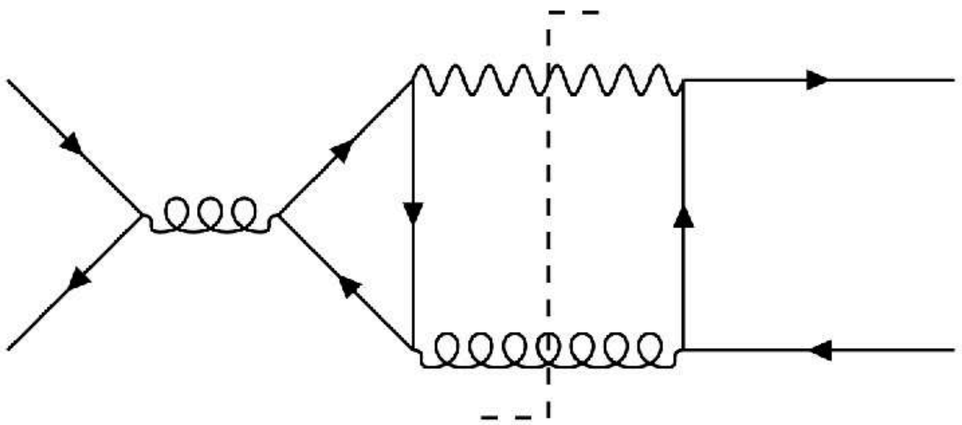


$$\delta_{f_a f_b} \sum_{f_1} c_{f_1 f_1}$$

Amplitudes squared and their EW couplings



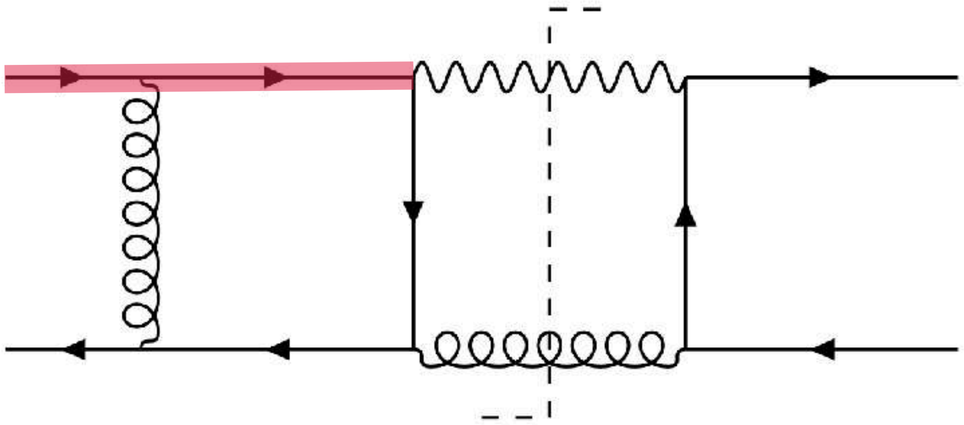
$$c_{f_a f_b} c'^{*}_{f_a f_b}$$



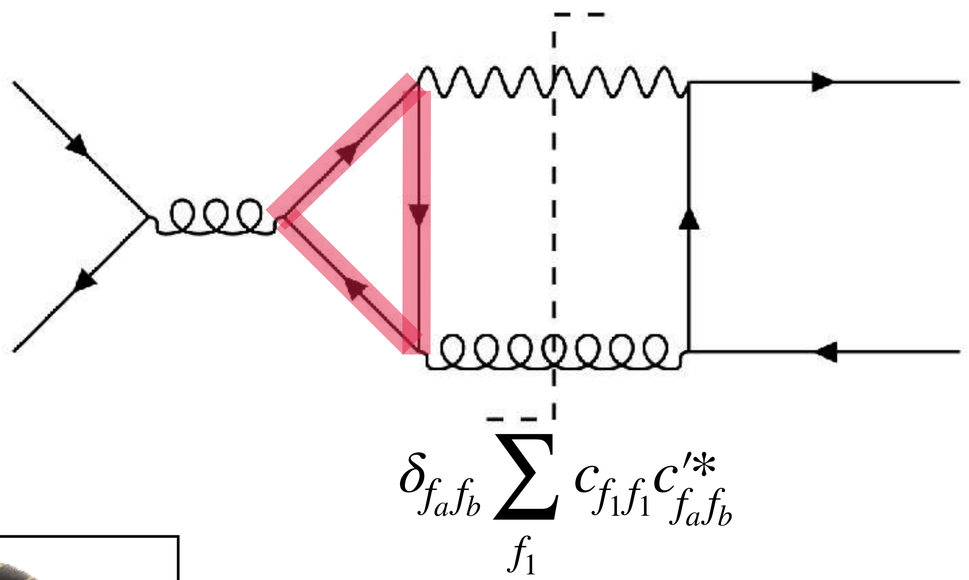
$$\delta_{f_a f_b} \sum_{f_1} c_{f_1 f_1} c'^{*}_{f_a f_b}$$

Pysanky diagrams

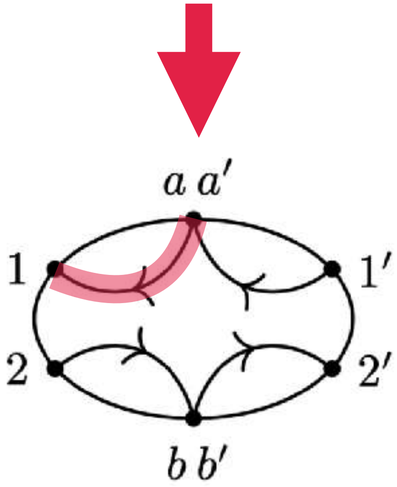
Diagrammatic way of organizing EW couplings in cross sections

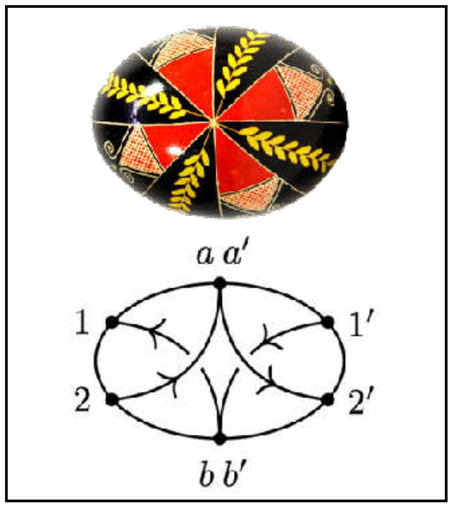


$$c_{f_a f_b} c'^{*}_{f_a f_b}$$

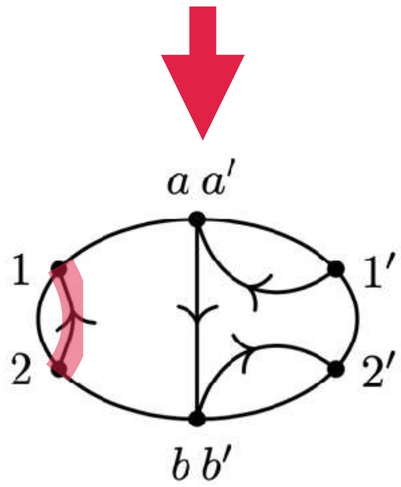


$$\delta_{f_a f_b} \sum_{f_1} c_{f_1 f_1} c'^{*}_{f_a f_b}$$





Pysanka



Conclusion

Motivation:
Helicity cross sections in DY: central to precision tests of QCD.

Ingredients:

- ✦ V+jet threshold factorization.
- ✦ RV extracted from massless spinor-helicity amplitudes.
- ✦ RR computed from scratch through conv. Feynman rules.
- ✦ Flavor decomposition through Easter egg diagrams.

Main result:
Computed NNLO helicity cross sections analytically at finite q_T.

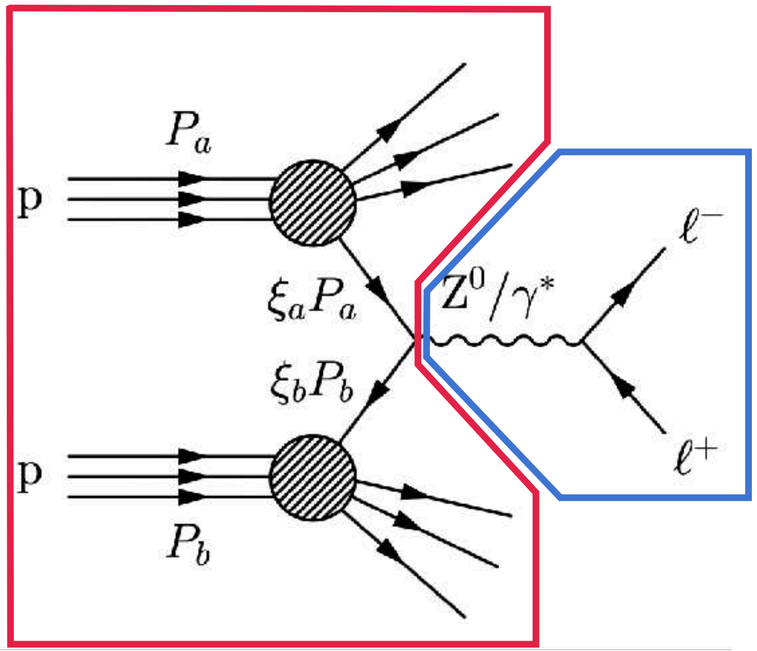
Outlook:
Get numerical results for PDF $\otimes \, \mathrm{d}\hat{\sigma}^{(2)}_{i,ab}/\mathrm{d}q_T$,
More usage for Z-helicity amplitudes, master integrals

Backup

Deriving angular coefficients

3 step plan

1. Differential cross section in terms of leptonic and hadronic tensors



$$\frac{\mathrm{d}\sigma(\Theta)}{\mathrm{d}^4 q\,\mathrm{d}\mathcal{O}} = \frac{1}{2E_{\mathrm{cm}}^2} L_{\mu\nu}(q, \mathcal{O}, \Theta) W^{\mu\nu}(q, P_a, P_b)$$

2. Axes and polarization vectors in CS frame:

$$t^\mu = \frac{q^\mu}{\sqrt{q^2}},$$

$$z^\mu = \frac{s_{bq} P_a^\mu - s_{aq} P_b^\mu}{\sqrt{s_{ab}\, s_{aq}\, s_{bq}}},$$

$$x^\mu = \frac{s_{aq}\, s_{bq}\, q^\mu - s_{bq}\, q^2\, P_a^\mu - s_{aq}\, q^2\, P_b^\mu}{\sqrt{s_{aq}\, s_{bq}\, q^2\, (s_{aq}\, s_{bq} - s_{ab}\, q^2)}},$$

$$y^\mu = \epsilon^{\mu\nu\rho\sigma} t_\nu x_\rho z_\sigma,$$

$$\Rightarrow \epsilon_\pm^\mu = \frac{1}{\sqrt{2}}(x^\mu \mp i y^\mu), \qquad \epsilon_0^\mu = z^\mu$$

3. Project the helicities of $W^{\mu\nu}$ onto the components of the $W_{\lambda\lambda'}$ and combine into structure functions W_i

$$W_{\lambda\lambda'} = \epsilon_\lambda^\mu \epsilon_{\lambda'}^{*\nu} W_{\mu\nu}$$

$$W_i = \sum_{\lambda,\lambda'=\text{comb.}} W_{\lambda\lambda'}$$

$$W_{-1} = W_{++} + W_{--},$$
$$W_0 = 2W_{00},$$
$$W_1 = -1/\sqrt{2}(W_{+0} + W_{0+} + W_{-0} + W_{0-}),$$
$$W_2 = -2(W_{+-} + W_{-+})$$
$$\cdots$$

Helicity cross sections

Relating to angular coefficients

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^4q\mathrm{d}\cos\theta\mathrm{d}\phi} = \frac{1}{2E_{\mathrm{cm}}^2}\frac{3}{16\pi}\sum_{i=-1}^{7} L_{\pm(i)}(q^2,\theta,\phi)W_i(q,P_a,P_b)g_i(\theta,\phi)$$

$$= \frac{3}{16\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}^4q}\left(1+\cos^2\theta+\frac{1}{2}A_0(1-3\cos^2\theta)+\sum_{i=1}^{7}A_i g_i(\theta,\phi)\right)$$

$$W_i = P_i^{\mu\nu}W_{\mu\nu}$$

$$\begin{aligned}
W_{-1} &= (x^\mu x^\nu + y^\mu y^\nu)W_{\mu\nu},\\
W_0 &= 2z^\mu z^\nu W_{\mu\nu},\\
W_1 &= -(x^\mu z^\nu + z^\mu x^\nu)W_{\mu\nu},\\
W_2 &= -2(x^\mu x^\nu - y^\mu y^\nu)W_{\mu\nu},\\
W_3 &= 2i(y^\mu z^\nu - z^\mu y^\nu)W_{\mu\nu},\\
W_4 &= 2i(x^\mu y^\nu - y^\mu x^\nu)W_{\mu\nu},\\
W_5 &= -(x^\mu y^\nu + y^\mu x^\nu)W_{\mu\nu},\\
W_6 &= -(y^\mu z^\nu + z^\mu y^\nu)W_{\mu\nu},\\
W_7 &= -2i(x^\mu z^\nu - z^\mu x^\nu)W_{\mu\nu}\,.
\end{aligned}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^4q} = \frac{\mathrm{d}\sigma_{-1}}{\mathrm{d}^4q} + \frac{1}{2}\frac{\mathrm{d}\sigma_0}{\mathrm{d}^4q}$$

Helicity cross sections:

$$\frac{\mathrm{d}\sigma_i}{\mathrm{d}^4q} = \frac{1}{2E_{\mathrm{cm}}^2}L_{\pm(i)}(q^2,\theta,\phi)W_i(q,P_a,P_b)$$

Angular coefficients:

$$A_i = \frac{\mathrm{d}\sigma_i}{\mathrm{d}\sigma_{-1} + 1/2\,\mathrm{d}\sigma_0}$$

Cross section and PDFs

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^4q\mathrm{d}\cos\theta\mathrm{d}\phi} = \frac{1}{2E_{\mathrm{cm}}^2}\frac{3}{16\pi}\sum_{i=-1}^{7} L_{\pm(i)}(q^2,\theta,\phi)W_i(q,P_a,P_b)g_i(\theta,\phi)$$

$$= \frac{3}{16\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}^4q}\Big(1+\cos^2\theta+\frac{1}{2}A_0(1-3\cos^2\theta)+\sum_{i=1}^{7}A_i g_i(\theta,\phi)\Big)$$

$$g_{-1}(\theta,\varphi) = 1+\cos^2\theta\,, \qquad g_2(\theta,\varphi) = \tfrac{1}{2}\sin^2\theta\cos(2\varphi)\,, \qquad g_5(\theta,\varphi) = \sin^2\theta\sin(2\varphi)\,,$$
$$g_0(\theta,\varphi) = 1-\cos^2\theta\,, \qquad g_3(\theta,\varphi) = \sin\theta\cos\varphi\,, \qquad g_6(\theta,\varphi) = \sin(2\theta)\sin\varphi\,,$$
$$g_1(\theta,\varphi) = \sin(2\theta)\cos\varphi\,, \qquad g_4(\theta,\varphi) = \cos\theta\,, \qquad g_7(\theta,\varphi) = \sin\theta\sin\varphi\,.$$

$$\frac{\mathrm{d}\sigma_i}{\mathrm{d}^4q} = \sum_{a,b}\int\frac{\mathrm{d}\xi_a}{\xi_a}\frac{\mathrm{d}\xi_b}{\xi_b} f_a^{[h_a]}(\xi_a,\mu_F^2) f_b^{[h_b]}(\xi_b,\mu_F^2)\hat{\sigma}_{i,ab}(\xi_a P_a,\xi_b P_b,q)$$

Helicity cross section

Parton distribution functions

Partonic cross section

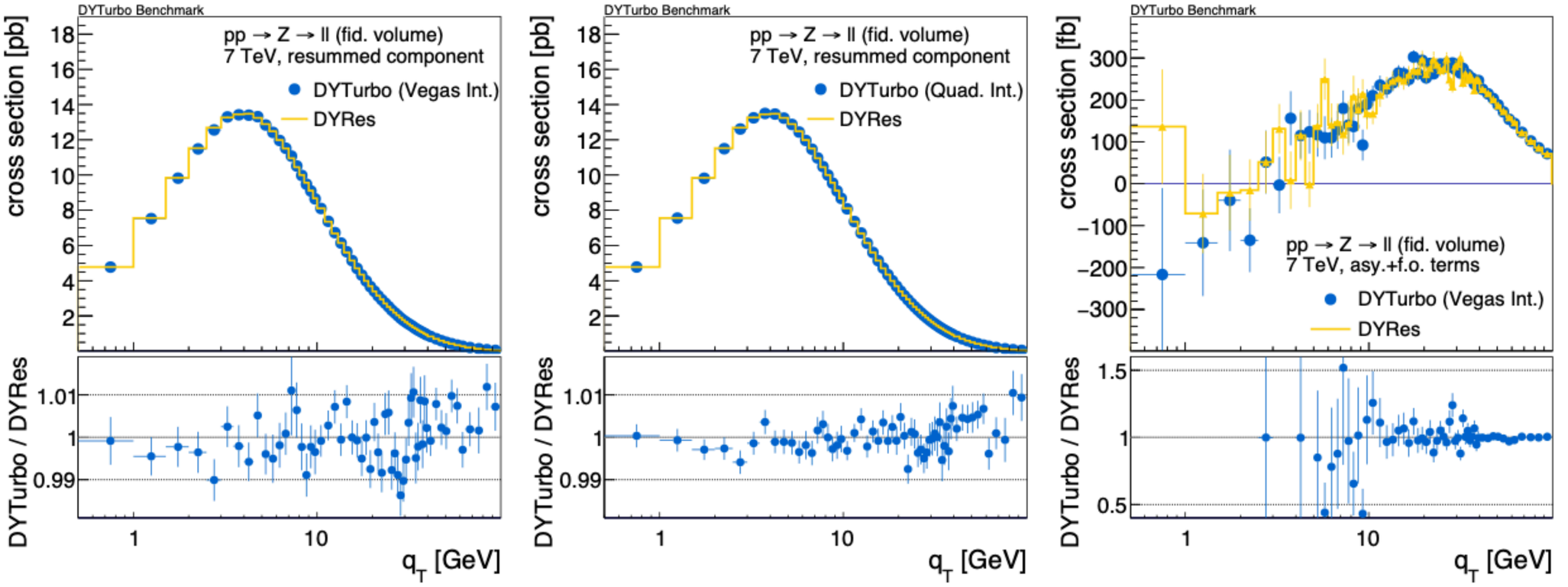
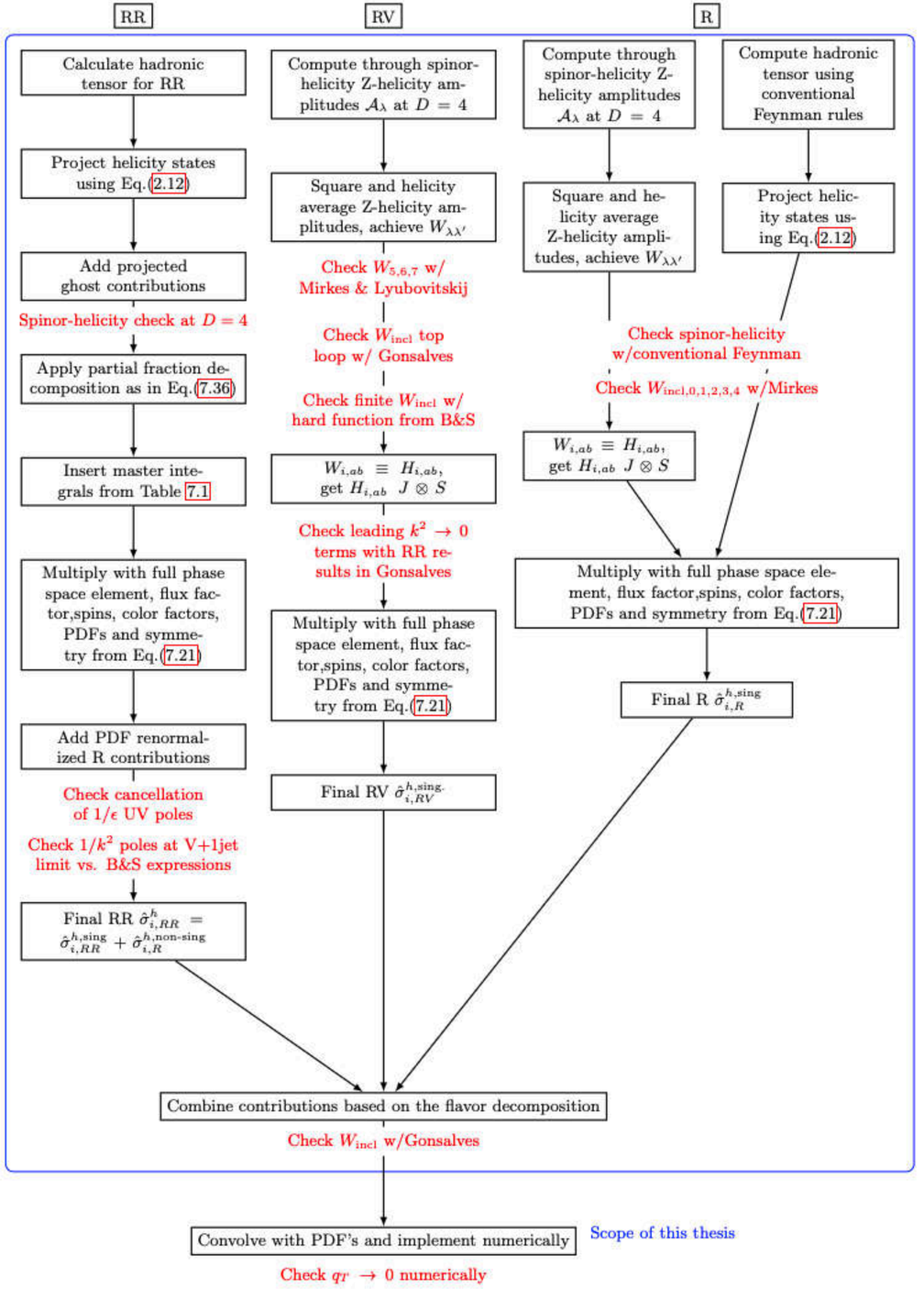


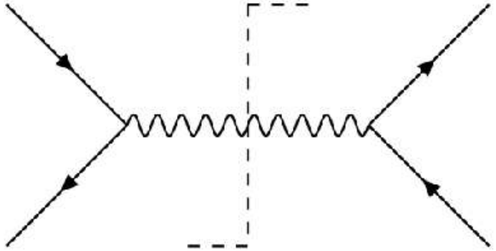
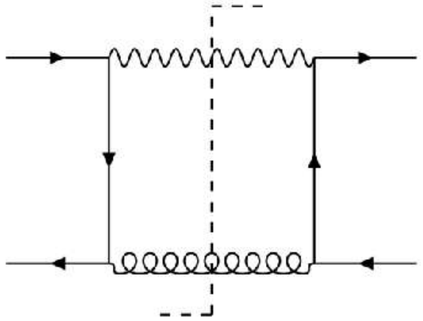
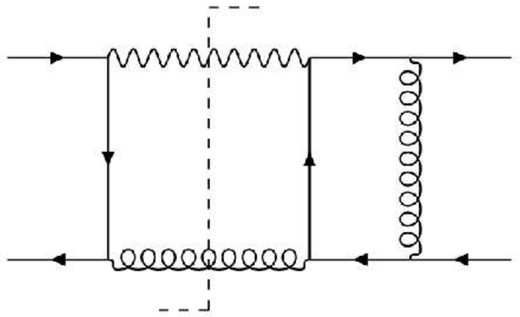
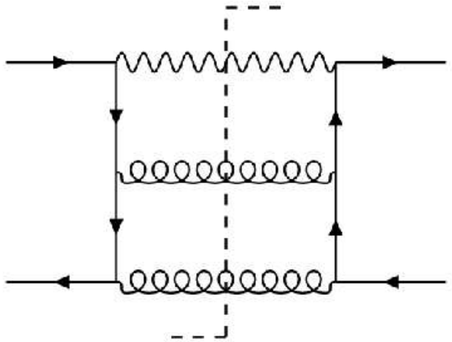
Figure 6 Comparison of differential fiducial cross sections computed with **DYRes** and **DYTurbo** at $\sqrt{s} = 7$ TeV as a function of the Z-boson transverse momentum. The Z-boson fiducial phase space is defined by the lepton transverse momentum $p_{\mathrm{T}}^{\ell} > 20$ GeV, the lepton pseudorapidity $|\eta_{\ell}| < 2.4$, and the invariant mass of the lepton pair 66 GeV $< m <$ 116 GeV. (a) Comparison of resummed component between **DYRes** and **DYTurbo** with Vegas integration. (b) Comparison of resummed component between **DYRes** and **DYTurbo** with Quadrature integration. (c) Comparison of the sum of asymptotic and finite-order terms between **DYRes** and **DYTurbo** with Vegas integration. The top panels show absolute cross sections, and the bottom panels show ratios of **DYTurbo** to **DYRes** results.

[S.Camarda, M.Boonekamp, G.Bozzi et al. '20]


RR
RV
R
Calculate hadronic tensor for RR
Project helicity states using Eq.(2.12)
Add projected ghost contributions
Spinor-helicity check at $D = 4$
Apply partial fraction decomposition as in Eq.(7.36)
Insert master integrals from Table 7.1
Multiply with full phase space element, flux factor,spins, color factors, PDFs and symmetry from Eq.(7.21)
Add PDF renormalized R contributions
Check cancellation of $1/\epsilon$ UV poles
Check $1/k^2$ poles at V+1jet limit vs. B&S expressions
Final RR $\hat{\sigma}^{h}_{i,RR} = \hat{\sigma}^{h,\text{sing}}_{i,RR} + \hat{\sigma}^{h,\text{non-sing}}_{i,R}$
Compute through spinor-helicity Z-helicity amplitudes $\mathcal{A}_\lambda$ at $D = 4$
Square and helicity average Z-helicity amplitudes, achieve $W_{\lambda\lambda'}$
Check $W_{5,6,7}$ w/ Mirkes & Lyubovitskij
Check W_{incl} top loop w/ Gonsalves
Check finite W_{incl} w/ hard function from B&S
$W_{i,ab} \equiv H_{i,ab}$, get $H_{i,ab}$ $J \otimes S$
Check leading $k^2 \to 0$ terms with RR results in Gonsalves
Multiply with full phase space element, flux factor,spins, color factors, PDFs and symmetry from Eq.(7.21)
Final RV $\hat{\sigma}^{h,\text{sing}}_{i,RV}$
Compute through spinor-helicity Z-helicity amplitudes $\mathcal{A}_\lambda$ at $D = 4$
Square and helicity average Z-helicity amplitudes, achieve $W_{\lambda\lambda'}$
Compute hadronic tensor using conventional Feynman rules
Project helicity states using Eq.(2.12)
Check spinor-helicity w/conventional Feynman
Check $W_{\text{incl},0,1,2,3,4}$ w/Mirkes
$W_{i,ab} \equiv H_{i,ab}$, get $H_{i,ab}$ $J \otimes S$
Multiply with full phase space element, flux factor,spins, color factors, PDFs and symmetry from Eq.(7.21)
Final R $\hat{\sigma}^{h,\text{sing}}_{i,R}$
Combine contributions based on the flavor decomposition
Check W_{incl} w/Gonsalves
Convolve with PDF's and implement numerically
Scope of this thesis
Check $q_T \to 0$ numerically

Radiative corrections

Perturbative expansion in strong coupling: $$\frac{\mathrm{d}\sigma}{\mathrm{d}q_T} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n \frac{\mathrm{d}\sigma^{(n)}}{\mathrm{d}q_T}$$

$\mathcal{O}(\alpha_s^0)$	$\mathcal{O}(\alpha_s^1)$	$\mathcal{O}(\alpha_s^2)$	
			
LO= Born × Born	NLO = $\mathrm{LO}_1 \times \mathrm{LO}_1$	NNLO = $\mathrm{LO}_1 \times \mathrm{NLO}_1$	NNLO = $\mathrm{LO}_2 \times \mathrm{LO}_2$
Born	R	RV	RR
$q\bar{q} \to V$	$q\bar{q} \to Vg, qg \to Vq$	$q\bar{q} \to Vg, qg \to Vq$	$q\bar{q} \to Vq\bar{q}, qq \to Vqq,$ $q\bar{q} \to Vgg, gg \to Vq\bar{q},$ $qg \to Vqg$
1 × 1 diagrams	2 × 2 diagrams per process	2 × 13 diagrams per process	8 × 8 diagrams per process

PDF renormalization

Master formula for PDF renormalization

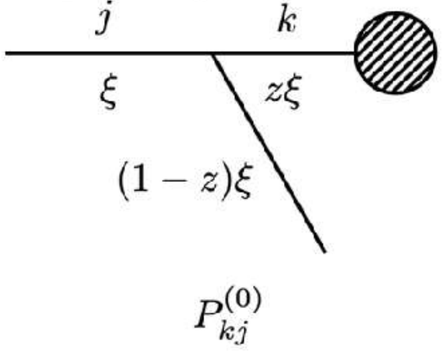
$$\hat{\sigma}^{(2)}_{i,ab}(\xi_a P_a, \xi_b P_b, q) = \hat{\sigma}^{\text{bare}\,(2)}_{i,ab}(\xi_a P_a, \xi_b P_b, q)$$

$a,\ b$ - incoming partons

$$+\frac{1}{\epsilon}\sum_{a'}\frac{P^{(0)}_{a'a}(z^*_a)}{\Delta^*_a}\bar{\sigma}^{\text{bare}\,(1)}_{i,a'b}(z^*_a p_a, p_b, q)$$

$$+\frac{1}{\epsilon}\sum_{b'}\frac{P^{(0)}_{b'b}(z^*_b)}{\Delta^*_b}\bar{\sigma}^{\text{bare}\,(1)}_{i,ab'}(p_a, z^*_b p_b, q)\,.$$

Splitting function



Achieved by performing the momentum fraction integral with $\delta(k^2)$ from NLO cross section first

Barred LO_1 xsection:

$$\hat{\sigma}^{\text{bare}\,(1)}_{i,a'b}(z_a p_a, p_b, q) = \bar{\sigma}^{\text{bare}\,(1)}_{i,a'b}(z_a p_a, p_b, q)\delta(k^2)\Big|_{k=z_a p_a + p_b - q}$$

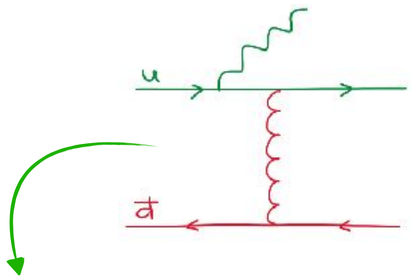
Starred mom. fraction and delta factor:

$$z^*_a = \frac{k^+ q^- + q_T^2}{k^+(k^- + q^-)}, \quad z^*_b = \frac{k^- q^+ + q_T^2}{k^-(k^+ + q^+)},$$

$$\Delta^*_a = k^+(k^- + q^-) \quad \Delta^*_b = k^-(k^+ + q^+),$$

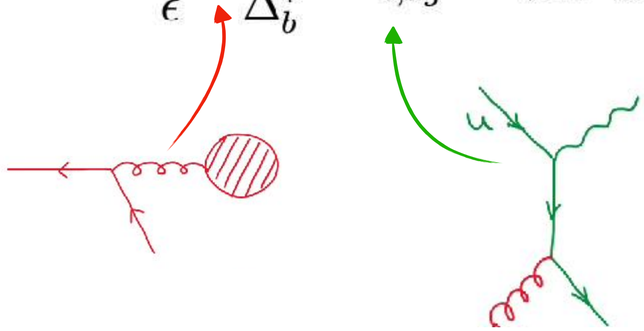
PDF renormalization

Master formula (T-channel $q\bar{q} \to Vq\bar{q}$)



$$\hat{\sigma}^{(2)}_{i,u\bar{d}}(p_a, p_b, q) = \hat{\sigma}^{(2),\,\text{bare}}_{i,u\bar{d}}(p_a, p_b, q)$$

$$+\frac{1}{\epsilon}\frac{P^{(0)}_{g\bar{d}}(z_b^*)}{\Delta_b^*}\bar{\sigma}^{(1),\,\text{bare}}_{i,ug}(p_a, z_b^* p_b, q)$$



V+jet threshold factorization as an analytic subtraction

Main idea

$$\hat{\sigma}^{\text{sing},(2)}_{i,ab} \propto \left[\hat{\sigma}^{\text{fin}}_{i,ab}\right]_{\text{RV}} \delta(k^2) + \left[\hat{\sigma}_{i,ab}\right]_{\text{R}} \left[\mathscr{L}_1(k^2) + \mathscr{L}_0(k^2) + \delta(k^2)\right]$$

Non-singular terms, since we are only left terms with RR:

$$\hat{\sigma}^{\text{non-sing},(2)}_{i,ab} = \left[\hat{\sigma}^{\text{full}}_{i,ab}\right]_{\text{RR}} - \left.\hat{\sigma}^{\text{sing},(2)}_{i,ab}\right|_{k^2>0}$$

- $\left[\hat{\sigma}^{\text{full},(2)}_{i,ab}\right]_{\text{RR}}$ - full PDF renormalized RR cross section
- $\left.\hat{\sigma}^{\text{sing},(2)}_{i,ab}\right|_{k^2>0} = \left[\hat{\sigma}_{i,ab}\right]_{\text{R}} \left[\dfrac{\ln(k^2/v^2)}{k^2} + \dfrac{1}{k^2}\right]$

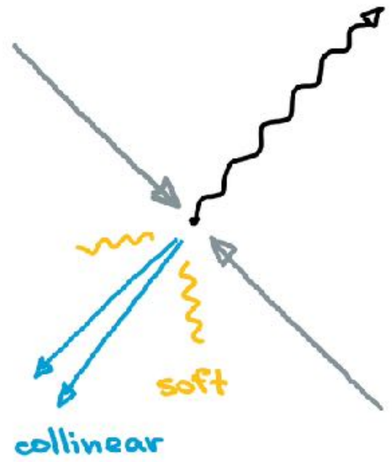
Two goals now: compute $\left[\hat{\sigma}^{\text{full}}_{i,ab}\right]_{\text{RR}}$ and $\left[\hat{\sigma}^{\text{fin}}_{i,ab}\right]_{\text{RV}}$

V+jet threshold factorization as an analytic subtraction

Main idea

Singular terms using SCET factorization formula:

$$\hat{\sigma}^{\text{sing}}_{i,ab} \propto H_i \, J \otimes S \to \hat{\sigma}^{\text{sing},(2)}_{i,ab} \propto H_i^{(2)} + H_i^{(1)} \left(J^{(1)} + S^{(1)} \right)$$



SCET factorization formula is fully finite in ϵ

$$\left[\frac{1}{\epsilon_{\text{UV}}} + \frac{a}{\epsilon_{\text{IR}}} + \frac{1}{\epsilon_{\text{IR}}^2} + \hat{\sigma}_{i,\text{fin}} \right]_{\text{RV}} \quad \left[\frac{b^{\text{sing}}}{\epsilon_{\text{IR}}} - \frac{1}{\epsilon_{\text{IR}}^2} + \hat{\sigma}_{\text{i,fin,sing}} \right]_{\text{RR}} \quad \left[-\frac{1}{\epsilon_{\text{UV}}} \right]_{\alpha_s\text{CT}} \quad \left[-\frac{a + b^{\text{sing}}}{\epsilon_{\text{IR}}} \right]_{\text{PDF CT}}$$

$$H^{(2)} \qquad + \quad H^{(1)} \left(J^{(1)} + S^{(1)} \right)$$

$$\hat{\sigma}^{\text{sing},(2)}_{i,ab} \propto \left[\hat{\sigma}^{\text{fin}}_{i,ab} \right]_{\text{RV}} \delta(k^2) + \left[\hat{\sigma}_{i,ab} \right]_{\text{R}} \left[\mathscr{L}_1(k^2) + \mathscr{L}_0(k^2) + \delta(k^2) \right]$$

Checks: spinor-helicity formalism

Main idea (first for $\mathbf{LO_1}$)

Factorize the spinor-helicity amplitude $p_1 p_2 \to p_3(V \to p_4 p_5)$

$$\mathscr{A}_{\text{full}}(1^{\lambda_g}, 2^{\lambda_q}_{\bar{q}}, 3^{-\lambda_q}_{q}, 4^{\lambda_\ell}_{\ell}, 5^{-\lambda_\ell}_{\bar{\ell}}) = \sum_{\lambda} \mathscr{A}_{\lambda}(1^{\lambda_g}, 2^{\lambda_q}_{\bar{q}}, 3^{-\lambda_q}_{q}) \, \mathscr{A}_{\ell\lambda}(1^{\lambda_g}, 2^{\lambda_q}_{\bar{q}}, 3^{-\lambda_q}_{q}, 4^{\lambda_\ell}_{\ell}, 5^{-\lambda_\ell}_{\bar{\ell}})$$

Z helicity amplitudes:

$$\mathscr{A}_{\lambda}(1^{\lambda_g}, 2^{\lambda_q}_{\bar{q}}, 3^{-\lambda_q}_{q}) = \{\mathscr{A}_0, \mathscr{A}_+, \mathscr{A}_-\}$$

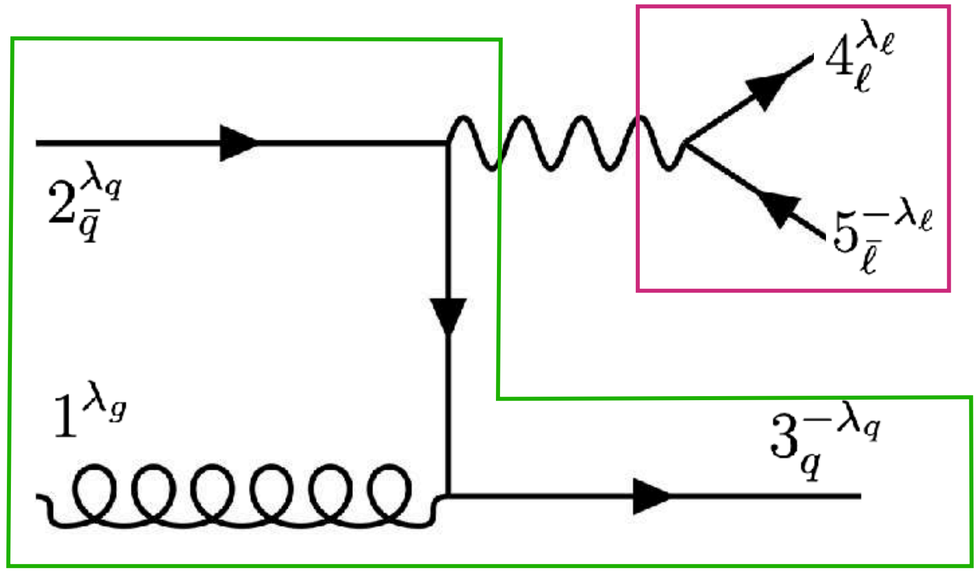
Projected leptonic current:

$$\mathscr{A}_{\ell\lambda}(1^{\lambda_g}, 2^{\lambda_q}_{\bar{q}}, 3^{-\lambda_q}_{q}, 4^{\lambda_\ell}_{\ell}, 5^{-\lambda_\ell}_{\bar{\ell}}) = \epsilon^{\mu}_{\lambda} \, L_{\lambda_\ell,\mu}(4^{\lambda_\ell}_{\ell}, 5^{-\lambda_\ell}_{\bar{\ell}})$$

Helicity projectors: $\epsilon^{\mu}_{\pm} = \frac{1}{\sqrt{2}}(x^\mu \mp i y^\mu), \qquad \epsilon^{\mu}_{0} = z^\mu$

Leptonic current:

$$\mathcal{M}_{\lambda\mu}(4^{\lambda_\ell}_{\ell}, 5^{-\lambda_\ell}_{\bar{\ell}}) \sim \begin{cases} [4|\gamma_\mu(v_\ell - a_\ell\gamma^5)|5\rangle, & \lambda_\ell = 1/2, \\ \langle 4|\gamma_\mu(v_\ell - a_\ell\gamma^5)|5], & \lambda_\ell = -1/2. \end{cases}$$



Goal: $W_{\lambda\lambda'} = \sum_{(\dots)} \mathcal{A}^*_{\lambda'}(\dots)\mathcal{A}_{\lambda}(\dots)$

polarisations of Z boson

summed over all helicities

Projected leptonic current:

$$\lambda_\ell > 0: \quad \mathcal{A}_{\ell 0} = (v_\ell - a_\ell)\frac{s_{1q}[24]\langle 25\rangle - s_{2q}[14]\langle 15\rangle}{\sqrt{s_{12}s_{1q}s_{2q}}},$$

$$\mathcal{A}_{\ell +} = (v_\ell - a_\ell)\left[\frac{[24]\langle 25\rangle s_{2q}\left(\sqrt{\frac{q^2 s_{12}}{s_{1q}s_{2q}}}+1\right) + [14]\langle 15\rangle s_{2q}\left(\sqrt{\frac{q^2 s_{12}}{s_{1q}s_{2q}}}-1\right) + 2\langle 25\rangle\langle 15\rangle[21][45]}{\sqrt{2}\sqrt{s_{12}}\sqrt{(s_{1q}s_{2q}-s_{12}q^2)}}\right],$$

$$\mathcal{A}_{\ell -} = (v_\ell - a_\ell)\left[\frac{[24]\langle 25\rangle s_{1q}\left(\sqrt{\frac{q^2 s_{12}}{s_{1q}s_{2q}}}-1\right) + [14]\langle 15\rangle s_{2q}\left(\sqrt{\frac{q^2 s_{12}}{s_{1q}s_{2q}}}+1\right) - 2\langle 25\rangle\langle 15\rangle[21][45]}{\sqrt{2}\sqrt{s_{12}}\sqrt{(s_{1q}s_{2q}-s_{12}q^2)}}\right].$$

$$\lambda_\ell < 0: \quad \mathcal{A}_{\ell 0} = (v_\ell + a_\ell)\frac{s_{1q}\langle 24\rangle[25] - s_{2q}\langle 14\rangle[15]}{\sqrt{s_{12}s_{1q}s_{2q}}},$$

$$\mathcal{A}_{\ell +} = (v_\ell + a_\ell)\left[\frac{\langle 24\rangle[25] s_{1q}\left(\sqrt{\frac{q^2 s_{12}}{s_{1q}s_{2q}}}-1\right) + \langle 14\rangle[15] s_{2q}\left(\sqrt{\frac{q^2 s_{12}}{s_{1q}s_{2q}}}+1\right) - 2[25][15]\langle 21\rangle\langle 45\rangle}{\sqrt{2}\sqrt{s_{12}}\sqrt{(s_{1q}s_{2q}-s_{12}q^2)}}\right],$$

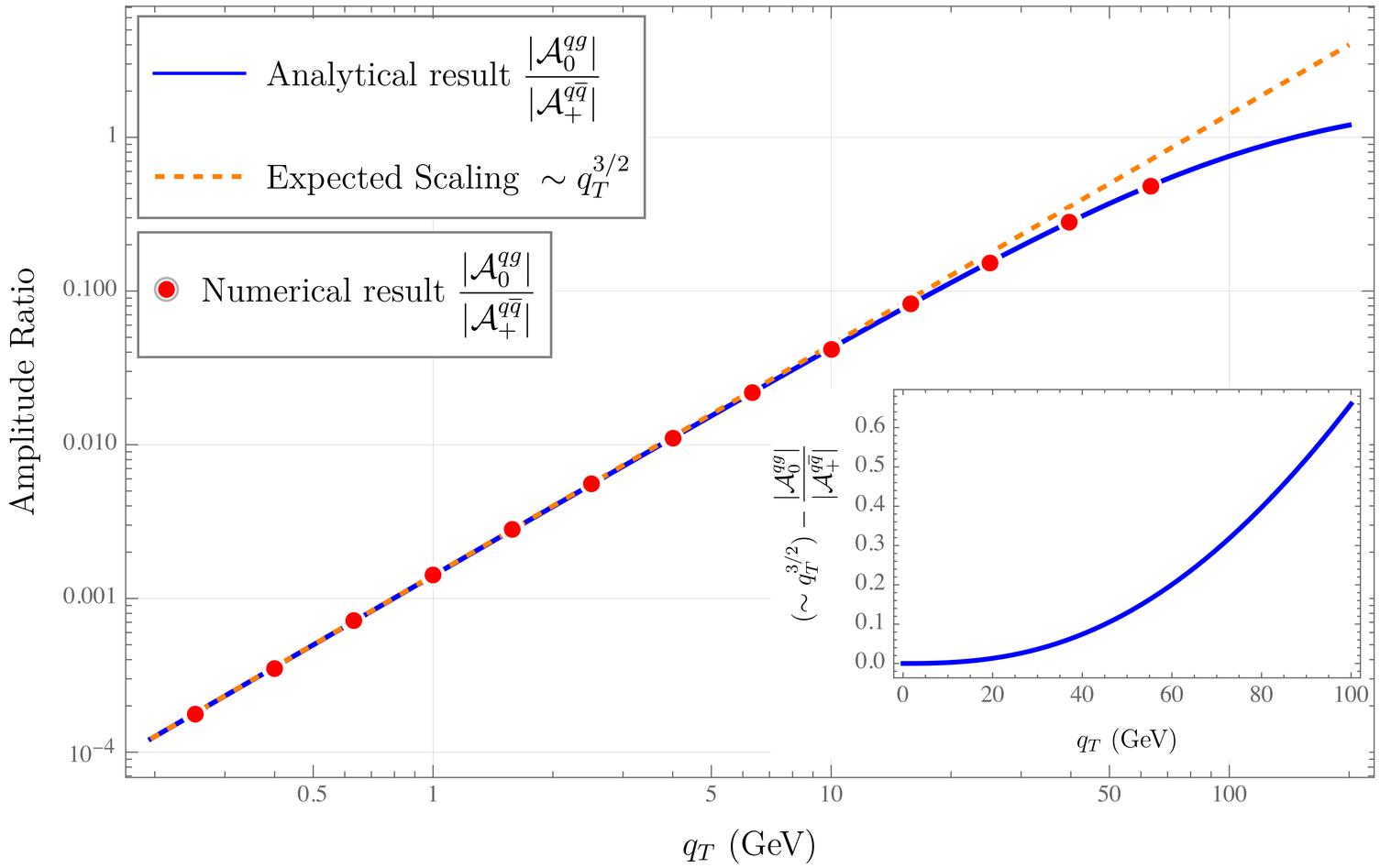
$$\mathcal{A}_{\ell -} = (v_\ell + a_\ell)\left[\frac{\langle 24\rangle[25] s_{1q}\left(\sqrt{\frac{q^2 s_{12}}{s_{1q}s_{2q}}}+1\right) + \langle 14\rangle[15] s_{2q}\left(\sqrt{\frac{q^2 s_{12}}{s_{1q}s_{2q}}}-1\right) + 2[25][15]\langle 21\rangle\langle 45\rangle}{\sqrt{2}\sqrt{s_{12}}\sqrt{(s_{1q}s_{2ggq}-s_{12}q^2)}}\right].$$

Power counting for Z-helicity amplitudes

Scaling in the low-q_T/Q limit

$$\lambda = \frac{q_T}{Q} \ll 1$$

$$|\mathscr{A}_0|_{qg\to Vq} \sim \mathcal{O}\left(\lambda^{1/2}\right),$$
$$|\mathscr{A}_+|_{q\bar{q}\to Vg} \sim \mathcal{O}\left(\lambda^{-1}\right).$$



Power counting for Z-helicity amplitudes

Scaling in the low-q_T/Q limit

$\lambda_g, \lambda_q, \lambda_\ell/\mathcal{A}_\lambda$	$\lvert\mathcal{A}_0\rvert$ scaling at soft limit	$\lvert\mathcal{A}_+\rvert$ scaling at soft limit	$\lvert\mathcal{A}_-\rvert$ scaling at soft limit
$\mathrm{LO}_1\ qg \to Zq$			
$+;+,\pm$	$\mathcal{O}\left(\lambda^{1/2}\right)$	$\mathcal{O}\left(\lambda^{1/2}\right)$	$\mathcal{O}\left(\lambda^{1/2}\right)$
$+;-,\pm$	$\mathcal{O}\left(\lambda^{1/2}\right)$	$\mathcal{O}\left(\lambda^{3/2}\right)$	$\mathcal{O}\left(1/\lambda^{1/2}\right)$
$\mathrm{LO}_1\ q\bar{q} \to Zg$			
$+;+,\pm$	$\mathcal{O}\left(\lambda^{0}\right)$	$\mathcal{O}\left(1/\lambda\right)$	$\mathcal{O}\left(\lambda\right)$
$+;-,\pm$	$\mathcal{O}\left(\lambda^{0}\right)$	$\mathcal{O}\left(\lambda\right)$	$\mathcal{O}\left(1/\lambda\right)$

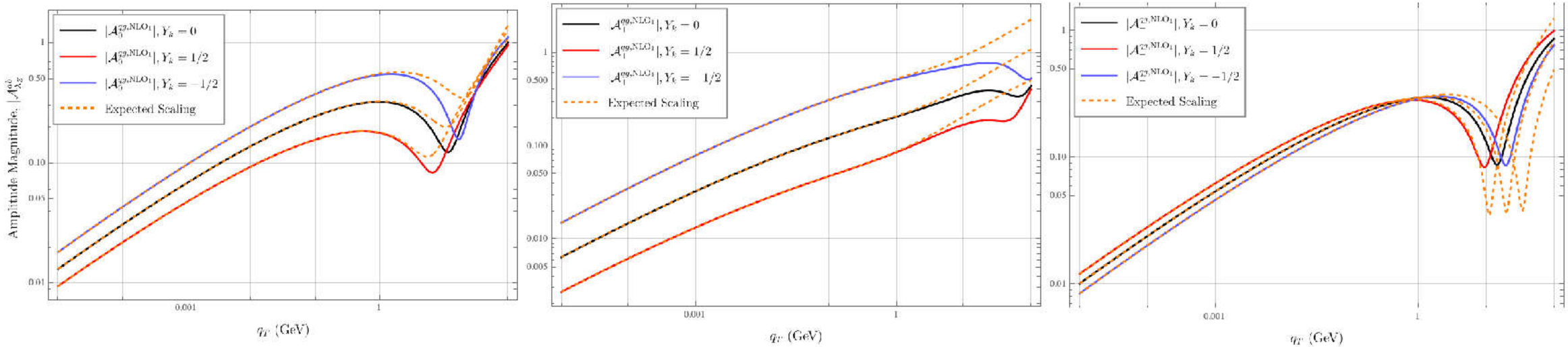
Power counting for Z-helicity amplitudes

$\lambda_Z = \{\ 0, +, -\ \}$

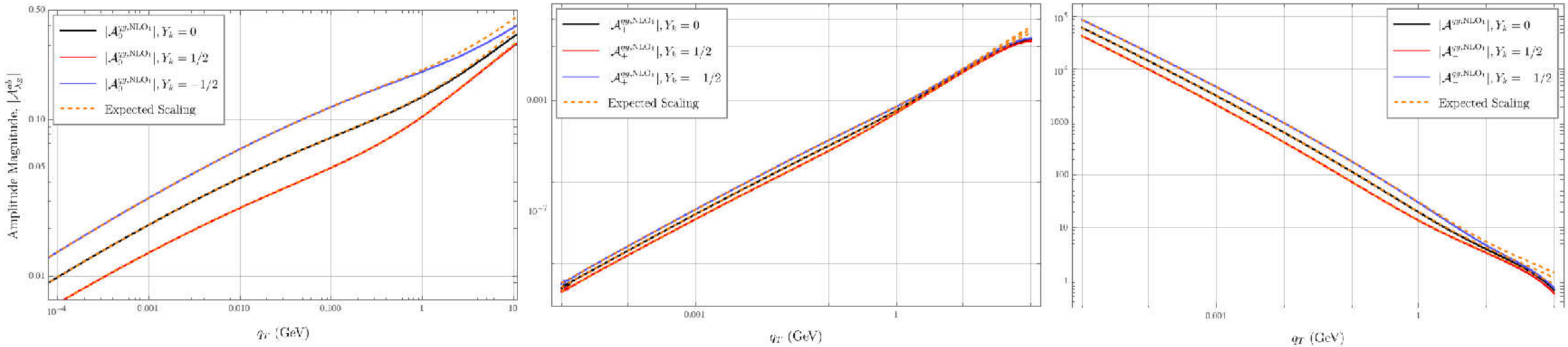
$\lambda_q = \{\ +, -\ \}$

$\mathrm{NLO}_1, qg \to Zq$

$\lambda_q > 0$

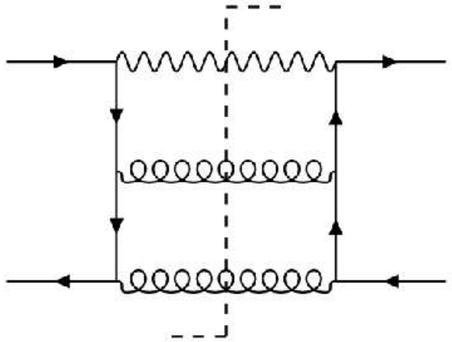


$\lambda_q < 0$



Main pipeline for RR

“Sausage factory” - boring (but important) stuff



Calculate hadronic tensor

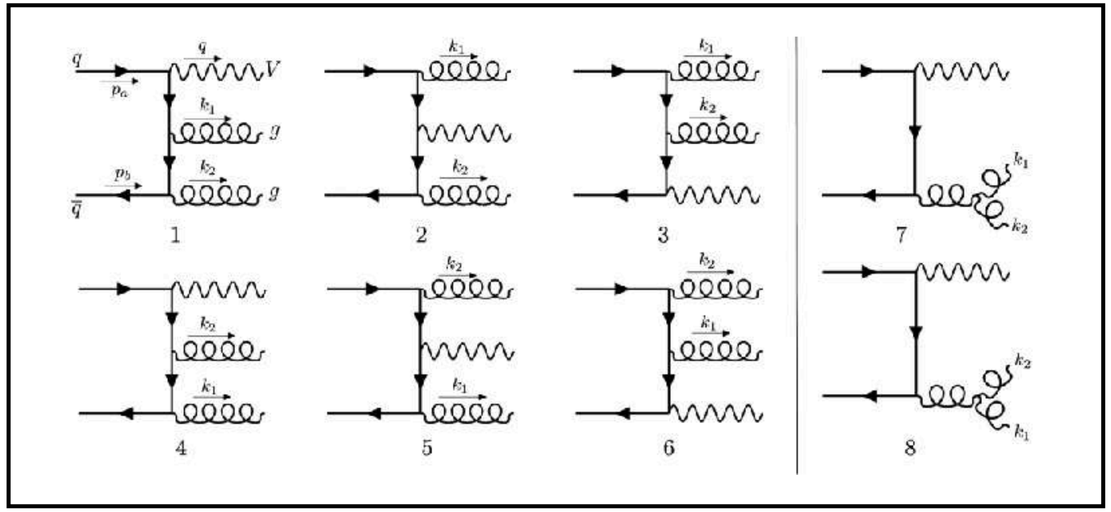
Project helicity states

Simplify the cross section

Apply partial fraction decomposition

Integrate over $k \to k_1 k_2$ decay space

Multiply with the rest of the factors + PDF CT



FeynCalc + FeynArts

[V.Shtabovenko, R.Mertig, F.Orellana '23] [T. Hahn '20]

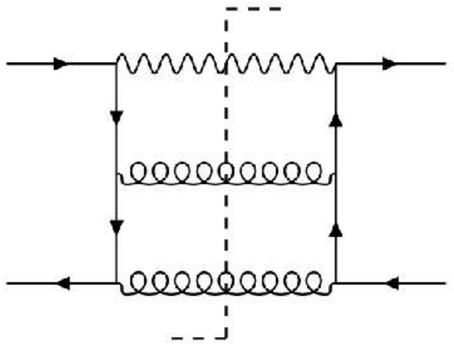
$$\Rightarrow W^{\mu\nu}(q, P_a, P_b)$$

Important detail: γ_5 scheme for RR in $D = 4 - 2\epsilon$ dimensions

- non-anticommuting γ_5
- $\gamma_\mu\gamma_5 = \frac{i}{6}\epsilon_{\mu\nu\rho\sigma}\gamma^\nu\gamma^\rho\gamma^\sigma$ [Larin '23] [Feyncalc manual]

Main pipeline for RR

“Sausage factory” - boring (but important) stuff



Calculate hadronic tensor

Project helicity states

Simplify the cross section

Apply partial fraction decomposition

Integrate over $k \to k_1 k_2$ decay space

Multiply with the rest of the factors + PDF CT

$$W_i(q, P_a, P_b) = \tilde{P}_i^{\mu\nu}(q, P_a, P_b)\, W^{\mu\nu}(q, P_a, P_b)$$

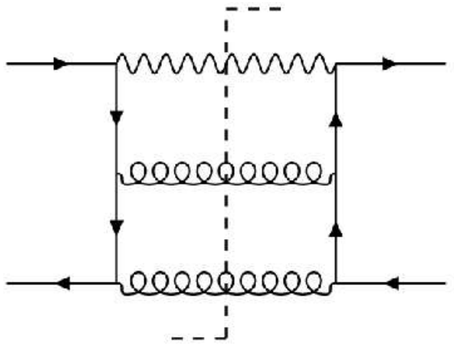
Structure functions

projectors

Hadronic tensor

Main pipeline for RR

"Sausage factory" - boring (but important) stuff



Calculate hadronic tensor

Project helicity states

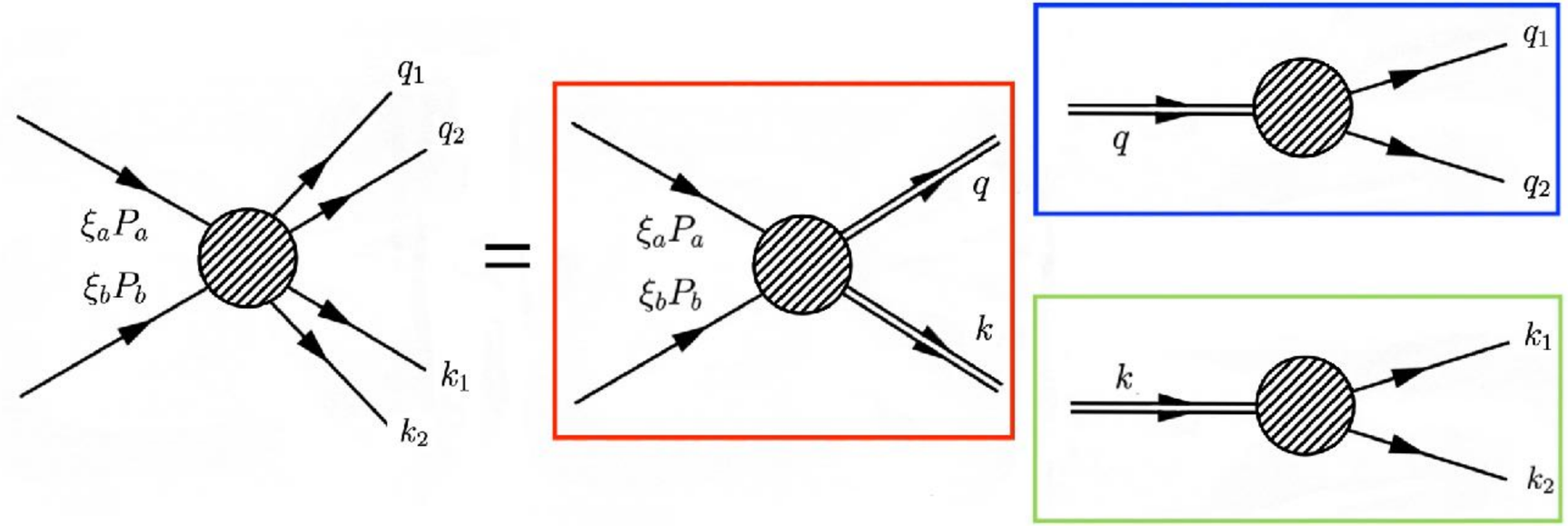
Simplify the cross section

Apply partial fraction decomposition

Integrate over $k \to k_1 k_2$ decay space

Multiply with the rest of the factors + PDF CT

$$\mathrm{d}\sigma_i^{\mathrm{bare}} \propto \int \mathrm{d}\phi_4 W_i, \quad \mathrm{d}\phi_4 \propto \mathrm{d}^D k_1 \, \mathrm{d}^D k_2 \, \mathrm{d}^D q_1 \, \mathrm{d}^D q_2$$



$$\mathrm{d}\phi_4 = \mathrm{d}\phi_{(p_a p_b \to qk)} \, \mathrm{d}\phi_{(q \to q_1 q_2)} \, \mathrm{d}\phi_{(k \to k_1 k_2)}$$

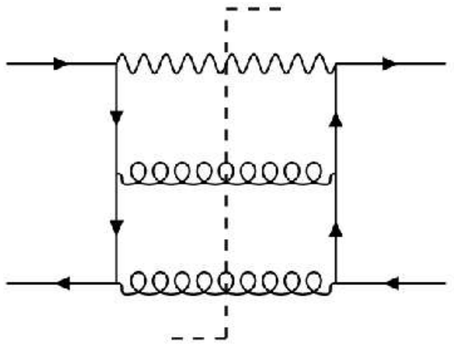
$\propto (\mathrm{d}k^- \mathrm{d}k^+ \mathrm{d}q^- \mathrm{d}q^+ \mathrm{d}q_T^2)$

$\propto \mathrm{d}\Omega^{(3-2\epsilon)}$ over lepton decay space untouched

$\propto \mathrm{d}\Omega^{(3-2\epsilon)}$ over k decay space integrated

Main pipeline for RR

"Sausage factory" - boring (but important) stuff



Calculate hadronic tensor

Project helicity states

Simplify the cross section

Apply partial fraction decomposition

Integrate over $k \to k_1 k_2$ decay space

Multiply with the rest of the factors + PDF CT

$$\mathrm{d}\sigma_i^{\text{bare}} \propto \int \mathrm{d}\phi_4 W_i, \quad \mathrm{d}\phi_4 \propto \mathrm{d}^D k_1 \, \mathrm{d}^D k_2 \, \mathrm{d}^D q_1 \, \mathrm{d}^D q_2$$

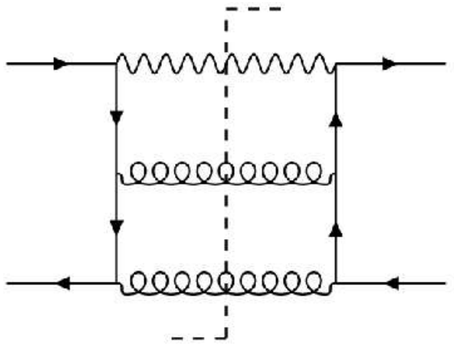
$$d\sigma^h_{i,ab} = d\phi_{p_a p_b \to q k} d\phi_{q \to q_1 q_2} \int d\phi_{k \to k_1 k_2} \frac{|M(p_a p_b \to q k_1 k_2)|^2_i}{2E^2_{cm}} \frac{f_a^{[h_a]}[\xi_a, \mu_F^2] f_b^{[h_b]}[\xi_b, \mu_F^2]}{n_a n_b N_a N_b} \times \text{sym.}$$

$$= \frac{(4\pi)^\epsilon}{(4\pi)^2 \Gamma(1-\epsilon)} \frac{(q_T^2)^{-\epsilon}}{2E^2_{cm}} \frac{dk^- dk^+ dq^- dq^+ dq_T^2}{(k^- + q^-)(k^+ + q^+)} \frac{f_a^{[h_a]}[\xi_a, \mu_F^2] f_b^{[h_b]}[\xi_b, \mu_F^2]}{n_a n_b N_a N_b} d\phi_{q \to q_1 q_2} \int \frac{d\phi_{k \to k_1 k_2}}{2\pi} |M|^2_i \times \text{sym.}$$

$$\times \theta(q^-)\theta(q^+)\theta(k^-)\theta(k^+)\theta(2E_a - k^- - q^-)\theta(2E_b - k^+ - q^+), \tag{7.21}$$

Main pipeline for RR

"Sausage factory" - boring (but important) stuff



Calculate hadronic tensor

Project helicity states

Simplify the cross section

Apply partial fraction decomposition

Integrate over $k \to k_1 k_2$ decay space

Multiply with the rest of the factors + PDF CT

Can have up to 4 unique denominators: $\int \frac{d\phi_{k\to k_1 k_2}}{2\pi} \frac{1}{(\text{den } 1)^{j_1}(\text{den } 2)^{j_2}(\text{den } 3)^{j_3}(\text{den } 4)^{j_4}}$

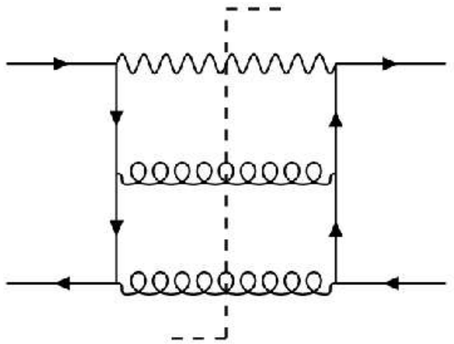
$$\frac{c_{12}}{(\text{den } 1)^{j_{12}}(\text{den } 2)^{j_{21}}} + \frac{c_{13}}{(\text{den } 1)^{j_{13}}(\text{den } 3)^{j_{31}}} + \frac{c_{14}}{(\text{den } 1)^{j_{14}}(\text{den } 4)^{j_{41}}} + \frac{c_{23}}{(\text{den } 2)^{j_{23}}(\text{den } 3)^{j_{32}}} + \frac{c_{24}}{(\text{den } 2)^{j_{24}}(\text{den } 4)^{j_{42}}} + \frac{c_{34}}{(\text{den } 3)^{j_{34}}(\text{den } 4)^{j_{43}}}$$

$$\frac{1}{ABCD} \to \frac{1}{ABC} + \frac{1}{ABD} + \frac{1}{ACD} + \frac{1}{BCD} \to \frac{1}{AB} + \frac{1}{AC} + \frac{1}{BC} \cdots$$

25

Main pipeline for RR

“Sausage factory” - boring (but important) stuff



Calculate hadronic tensor

Project helicity states

Simplify the cross section

Apply partial fraction decomposition

Integrate over $k \to k_1 k_2$ decay space

Multiply with the rest of the factors + PDF CT

In general $\mathrm{d}\phi_{(k\to k_1 k_2)}$ has non-trivial dependence on k_1, k_2 momenta.

Our master integrals help integrate D dim. expressions like

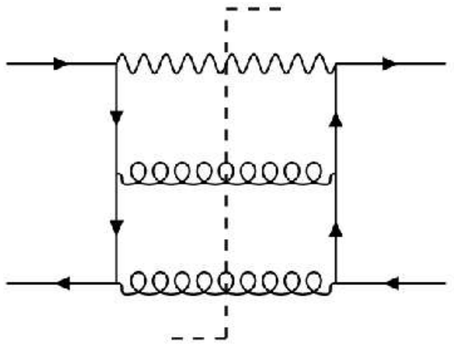
$$\int \frac{\mathrm{d}\phi_{(k\to k_1 k_2)}}{[(p_a-k_1)^2]^{j_1}[(p_b-k_2)^2]^{j_2}} \propto \int \frac{\mathrm{d}\Omega^{(3-2\epsilon)}}{(1-\beta_1\cos\theta_2\sin\theta_1-\beta_2\cos\theta_1)^{j_1}(1-\beta_3\cos\theta_2\sin\theta_1-\beta_4\cos\theta_1)^{j_2}}$$

[G.Somogyi '11]

Result: set of 6 unique master integrals.

Main pipeline for RR

"Sausage factory" - boring (but important) stuff



Calculate hadronic tensor

Project helicity states

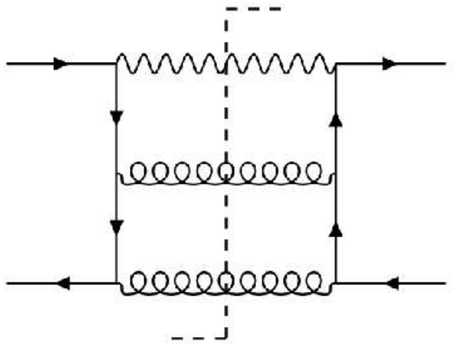
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Multiply with the rest of the factors + PDF CT

	$(p_a - k_1)^2$	$(p_a - k_2)^2$	$(p_b - k_1)^2$	$(p_b - k_2)^2$	$(q + k_1)^2$	$(q + k_2)^2$
$(p_a - k_1)^2$	One den, massless (7.28)	Separable	Two den's, both massless (7.32a)	Two den's, both massless (7.32b)	Two den's, one massive (7.33a)	Two den's, one massive (7.33b)
$(p_a - k_2)^2$	Separable	One den, massless (7.28)	Two den's, both massless (7.32b)	Two den's, both massless (7.32a)	Two den's, one massive (7.33b)	Two den's, one massive (7.33a)
$(p_b - k_1)^2$	Two den's, both massless (7.32a)	Two den's, both massless (7.32b)	One den, massless (7.29)	Separable	Two den's, one massive (7.33c)	Two den's, one massive (7.33d)
$(p_b - k_2)^2$	Two den's, both massless (7.32b)	Two den's, both massless (7.32a)	Separable	One den, massless (7.29)	Two den's, one massive (7.33d)	Two den's, one massive (7.33c)
$(q + k_1)^2$	Two den's, one massive (7.33a)	Two den's, one massive (7.33b)	Two den's, one massive (7.33c)	Two den's, one massive (7.33d)	One den, massive (7.30)	Separable
$(q + k_2)^2$	Two den's, one massive (7.33b)	Two den's, one massive (7.33a)	Two den's, one massive (7.33d)	Two den's, one massive (7.33c)	Separable	One den, massive (7.30)



Main pipeline for RR

"Sausage factory" - boring (but important) stuff

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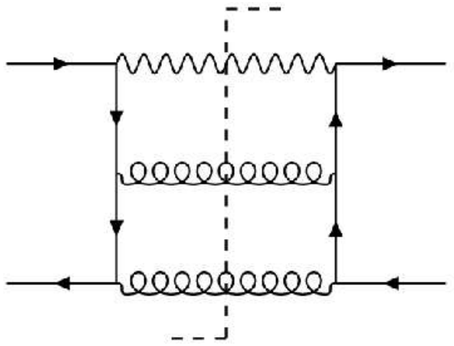
Multiply with the rest of the factors + PDF CT

$$\int \frac{d\phi_{k\to k_1k_2}}{2\pi} \frac{1}{\left[(q+k_1)^2\right]^{j_1}\left[(p_b-k_2)^2\right]^{j_2}} = \int \frac{d\phi_{k\to k_1k_2}}{2\pi} \frac{1}{\left[(q+k_2)^2\right]^{j_1}\left[(p_b-k_1)^2\right]^{j_2}} =$$

$$(-1)^{j_1+j_2}\frac{(4\pi)^\epsilon}{(k^+k^- - q_T^2)^\epsilon}\left[-\frac{(k^-+q^-)(k^-q^+ + q_T^2)}{k^-}\right]^{-j_1}\left[k^-(k^++q^+)\right]^{-j_2}\frac{1}{(4\pi)^2}\frac{\Gamma(1-j_2-\epsilon)}{\Gamma(2-j_2-2\epsilon)}$$

$$\times F_1\left(j_1, 1-j_2-\epsilon, 1-j_2-\epsilon, 2-j_2-2\epsilon, 1-\frac{k^-\left(\sqrt{4\left(k^-+q^-\right)\left(k^++q^+\right)q_T^2+\left(k^+q^- - k^-q^+\right)^2}+k^-q^++q^-\left(k^++2q^+\right)\right)}{2\left(k^-+q^-\right)\left(k^-q^++q_T^2\right)}\right.$$

$$\left. 1-\frac{k^-\left(-\sqrt{4\left(k^-+q^-\right)\left(k^++q^+\right)q_T^2+\left(k^+q^- - k^-q^+\right)^2}+k^-q^++q^-\left(k^++2q^+\right)\right)}{2\left(k^-+q^-\right)\left(k^-q^++q_T^2\right)}\right).$$



Calculate hadronic tensor

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Main pipeline for RR

"Sausage factory" - boring (but important) stuff

$$\sigma_i^{\text{full},(2)} \sim (\mathrm{d}k^- \mathrm{d}k^+ \mathrm{d}q^- \mathrm{d}q^+ \mathrm{d}q_T) \quad \text{2-to-2 measure}$$

$$\times \mathrm{d}\phi_{(q \to q_1 q_2)} \quad \text{lepton decay phase space}$$

$$\times \int \mathrm{d}\phi_{(k \to k_1 k_2)} |\mathscr{M}|^2_{i,ab} \quad \text{k decay space and matrix el.}$$

$$\times f_a^{[h_a]} f_b^{[h_b]} \quad \text{PDF's}$$

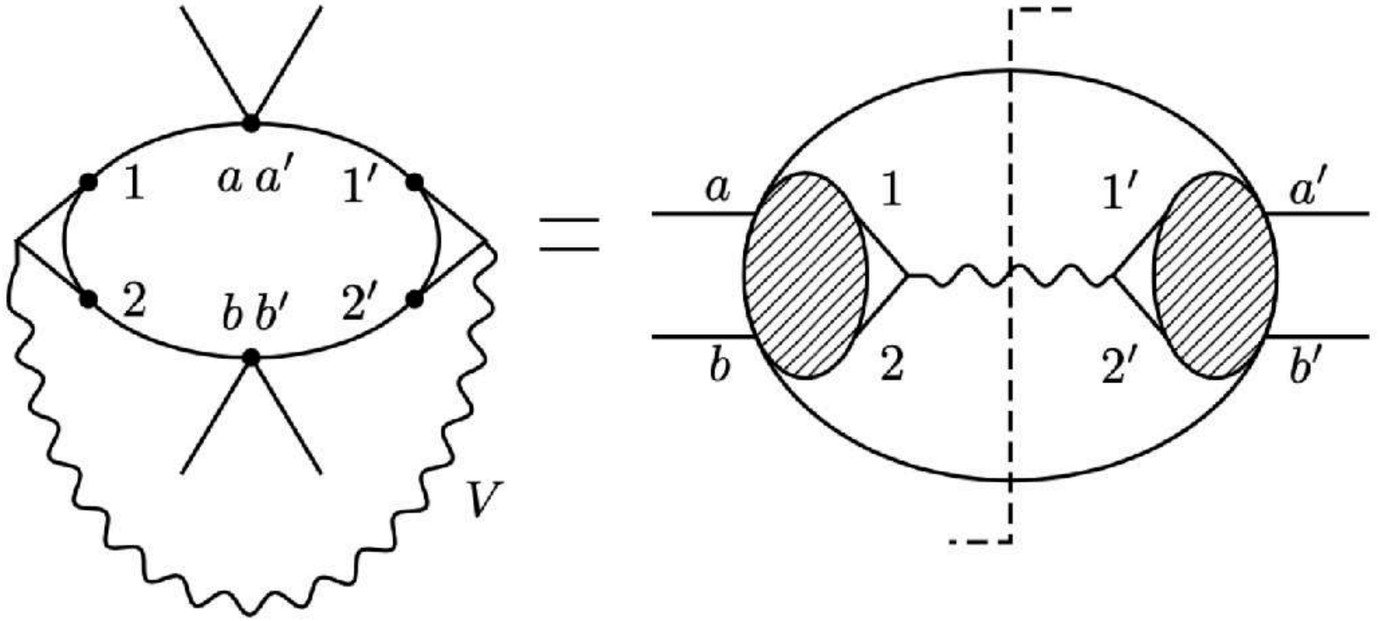
$$\times \text{fac}\,. \quad \text{flux, symmetry, spin, color etc.}$$

$$+ \frac{1}{\epsilon} \sum \frac{P^{(0)}_{\dots}(z^*)}{\Delta^*} \bar{\sigma}^{\text{bare},(1)}_{i,a'b} \quad \text{PDF renormalization}$$

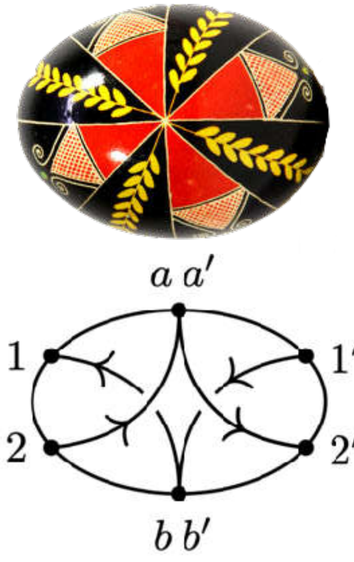
Putting results together

Flavor decomposition - Easter Egg diagrams

But! We calculated everything for $q = u$, $\bar{q} = \bar{u}$, what about different quarks?



[PYSANKA MOSAIC web]



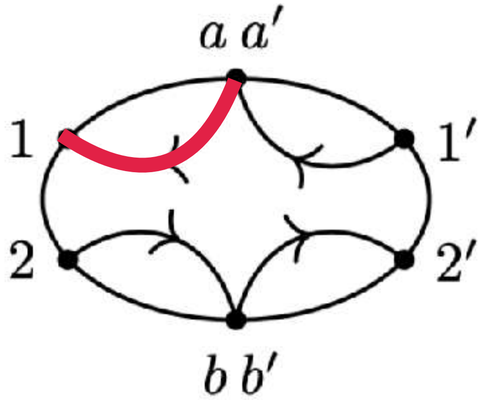
Pysanka

$$\prod_{\text{flavor } i,j} \delta_{ij} \times c_{f_1 f_2} c'^{*}_{f'_1 f'_2} \qquad c_{f_1 f_2} = \begin{cases} -iQ_f e, \text{ where } f = f_1 = f_2, \text{ for } \gamma^*, \\ \pm \frac{g_W}{2\sqrt{2}} V_{f_2 f_1}, \text{ where } f_1 \neq f_2, \text{ for } W^{\pm}, \\ \pm i \frac{g_Z}{2} v_f \text{ or } \pm i \frac{g_Z}{2} a_f, \text{ where } f = f_1 = f_2, \text{ for } Z. \end{cases}$$

Goal of the approach: get the exact number of unique flavor couplings

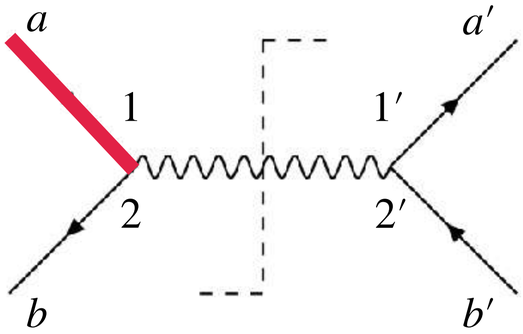
Putting results together

Flavor decomposition - Easter Egg diagrams



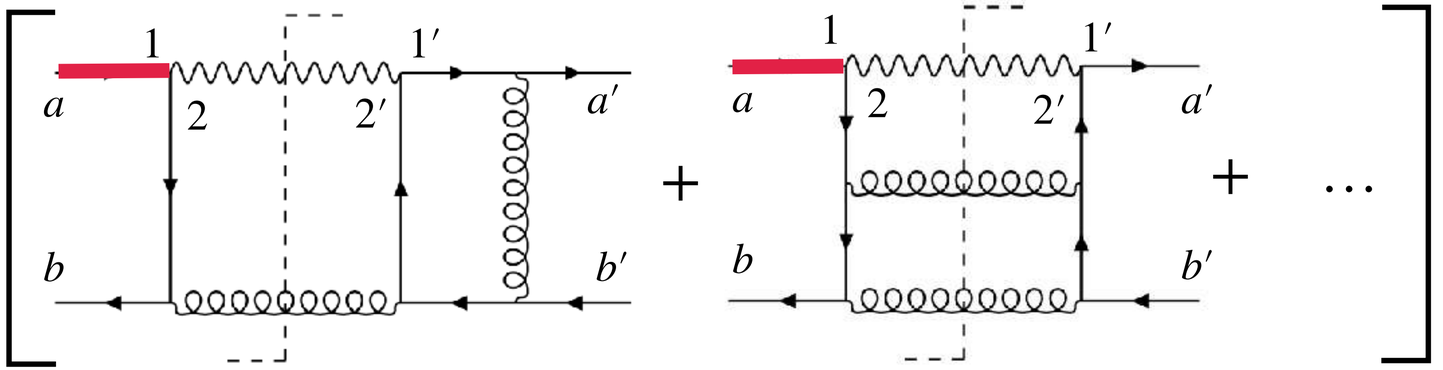
$c_{f_a f_b} c'^{*}_{f_a f_b}$

$= \mathcal{O}(\alpha_s^0)$



$+ \quad \mathcal{O}(\alpha_s^1)\ldots$

$+ \mathcal{O}(\alpha_s^2)$



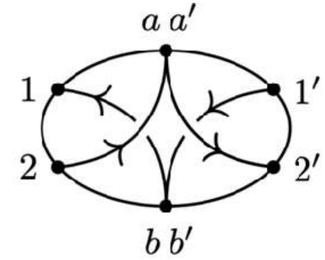
At $\mathcal{O}(\alpha_s^2)$ $p_a = g, p_b = g$: 2 comb. of EW couplings

$p_a = q, p_b = g$: 2 …

$p_a = q, p_b = \bar{q}$: 8 …

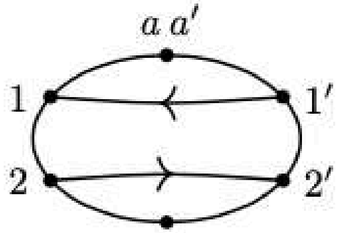
[PYSANKA MOSAIC web]



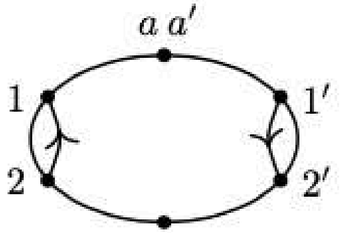


Pysanka

$gg \to VX$

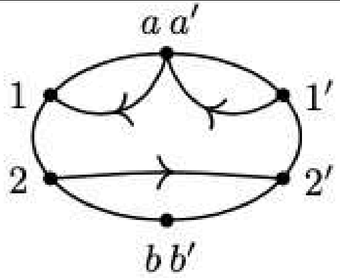


A1

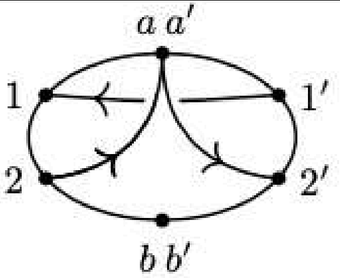


A2

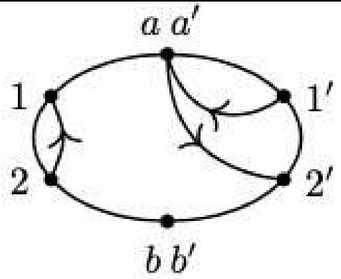
$gg \to VX$



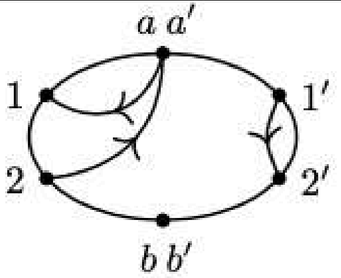
B1



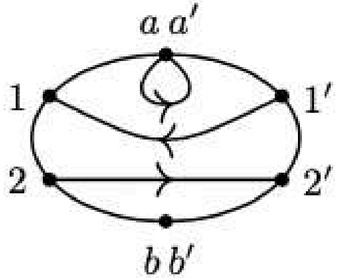
B2



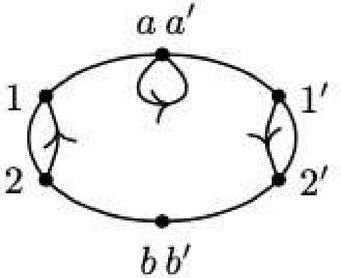
B3



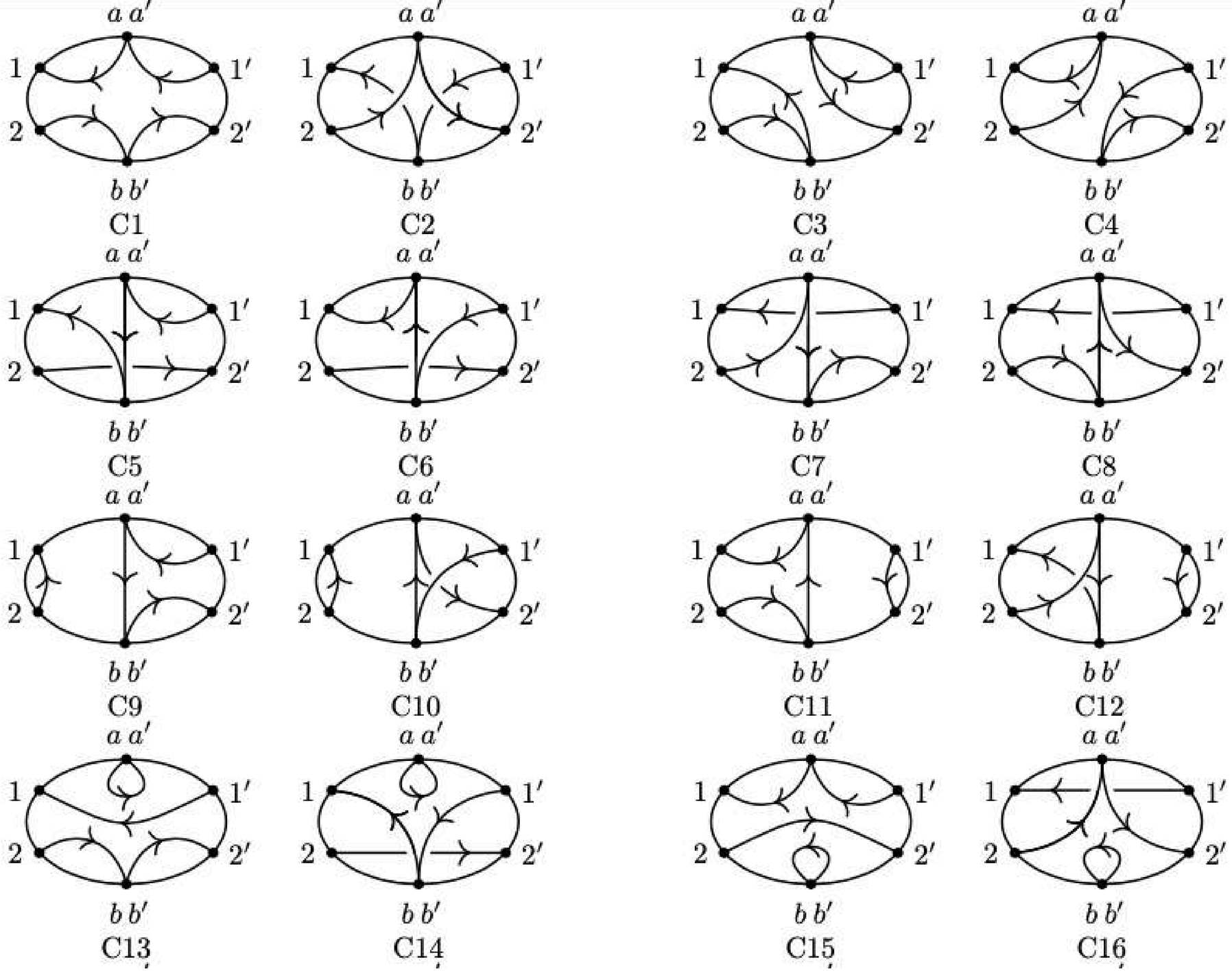
B4



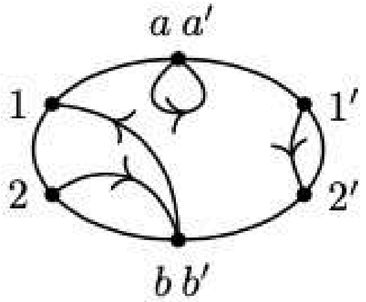
B5



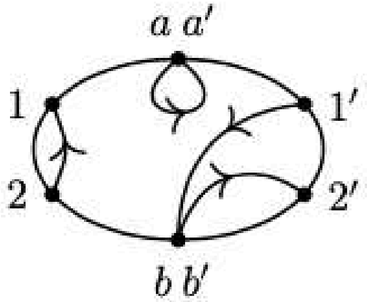
B6



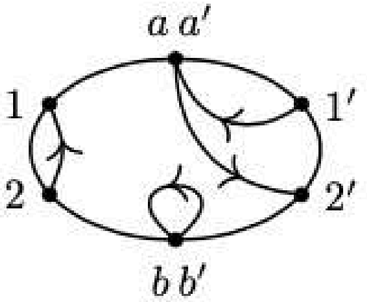
$X \wedge \leftarrow \underline{b}b$



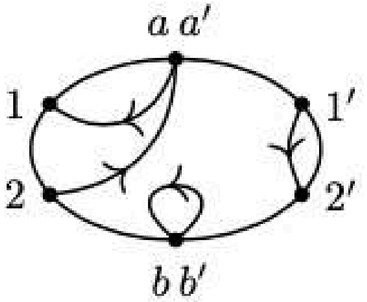
C17



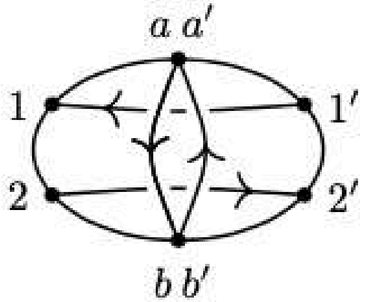
C18



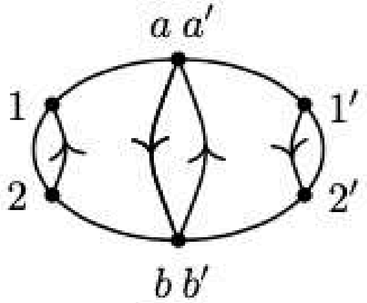
C19



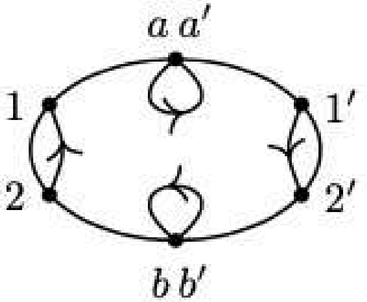
C20



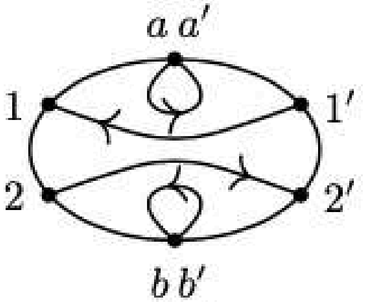
C21



C22



C23



C24

Extra extra

$$[f(x)]_+^{[x_0]} = \lim_{\epsilon\to 0}\left[f(x)\theta(x-\epsilon) + \delta(x-\epsilon)\int_{x_0}^{x} dy\, f(y)\right]$$

$$\mathcal{L}_n(x) \equiv \left[\frac{\theta(x)\ln^n x}{x}\right]_+ = \lim_{\epsilon\to 0}\left[\frac{\theta(x-\epsilon)\ln^n x}{x} + \delta(x-\epsilon)\frac{\ln^{n+1}\epsilon}{n+1}\right], \text{ with } n \geq 0,$$

DGLAP

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} f_k^{[h]}(\xi,\mu_F^2) = \sum_{n=0}^{\infty}\left(\frac{\alpha_s(\mu_F^2)}{4\pi}\right)^{n+1}\int_\xi^1 \frac{dz}{z}\sum_{j=1}^{2n_f+1} f_j^{[h]}\left(\frac{\xi}{z},\mu_F^2\right) P_{kj}^{(n)}(z)$$

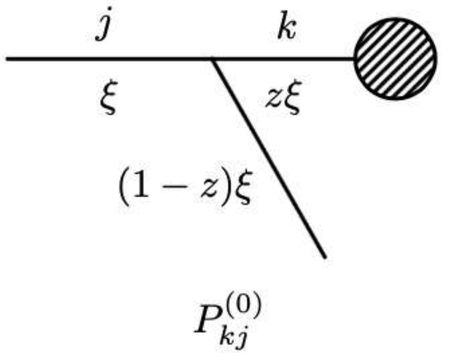


Figure D.1: LO splitting $j \to k$ with momentum fractions $\xi \to (z\xi, (1-z)\xi)$, described by $P_{kj}^{(0)}(z)$.

$$P_{qq}^{(0)}(z) = 2C_F\theta(z)P_{qq}(z), \quad P_{qq}(z) = \left(2\mathcal{L}_0(1-z) - \theta(1-z)(1+z) + \frac{3}{2}\delta(1-z)\right) = \left[\theta(1-z)\frac{1+z^2}{1-z}\right]_+,$$

$$P_{gq}^{(0)}(z) = 2C_F\theta(z)P_{gq}(z), \quad P_{gq}(z) = \theta(1-z)\frac{1+(1-z)^2}{z},$$

$$P_{qg}^{(0)}(z) = 2T_F\theta(z)P_{qg}(z), \quad P_{qg}(z) = \theta(1-z)[1-2z(1-z)],$$

$$P_{gg}^{(0)}(z) = 2C_A\theta(z)P_{gg}(z) + \beta_0\delta(1-z), \quad P_{gg}(z) = 2\mathcal{L}_0(1-z) + \theta(1-z)\left[2z(1-z) + \frac{2(1-z)}{z} - 2\right],$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$ is the leading-order QCD beta function coefficient.