Section 1: Lorentz Invariance

Wednesday, September 9th 2026

Srednicki, Ch. 2 + Schwartz Ch. 2

Conventions

We set $c = \hbar = 1$, write $x^\mu = (t, \vec{x})$, and use Srednicki's mostly-plus metric

$$g_{\mu\nu} = \text{diag}(-1, +1, +1, +1), \qquad x^2 \equiv x_\mu x^\mu = -t^2 + \vec{x}^{\,2}.$$

Greek indices run over $0, 1, 2, 3$, Latin spatial indices run over $1, 2, 3$, and any index that is repeated once upstairs and once downstairs is summed over. The opposite metric convention is equally valid, but mixing the two conventions inside a single calculation is not.

Raising and lowering indices. The metric is what lets us move an index from one position to the other,

$$x_\mu = g_{\mu\nu} x^\nu, \qquad x^\mu = (t, x, y, z), \qquad x_\mu = (-t, x, y, z).$$

Moving a *time* index up or down therefore flips the sign, whereas moving a spatial index does not.

The Einstein summation convention. The rule that an index repeated once upstairs and once downstairs is automatically summed over is called the *Einstein summation convention*, and it is what lets the same dot product be written in any of the following ways,

$$a_\mu b^\mu = a^\mu b_\mu = g_{\mu\nu} a^\mu b^\nu = g^{\mu\nu} a_\mu b_\nu = \sum_{\mu=0}^{3} a_\mu b^\mu = a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3.$$

Here $g^{\mu\nu}$ is the inverse metric, which raises an index in the same way that $g_{\mu\nu}$ lowers one. The two operations undo each other,

$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu,$$

where $\delta^\mu{}_\nu$ is the *Kronecker delta* and acts as the identity on indices. For the mostly-plus metric the inverse happens to have the same entries as the metric itself.

1 Why Lorentz transformations come first

Inertial observers disagree about the time and the position of an event, but they must agree about certain physical statements. They must agree about whether two events can causally influence one another, about the mass of a particle, and about the value of the action. *Lorentz transformations are exactly the linear changes of inertial coordinates that preserve these statements.*

2 Spacetime and the invariant interval

For two events separated by $\Delta x^\mu = (\Delta t, \Delta \vec{x})$, the invariant interval is

$$\Delta s^2 = g_{\mu\nu} \Delta x^\mu \Delta x^\nu = -(\Delta t)^2 + |\Delta \vec{x}|^2.$$

All inertial observers obtain the same value of Δs^2.

The interval plays the same role in spacetime that ordinary distance plays in Euclidean geometry. A rotation in the x–y plane changes x and y separately but preserves $x^2 + y^2$, and in the same way a Lorentz transformation mixes t with a spatial coordinate but preserves $-t^2 + \vec{x}^{\,2}$. Because the sign of the interval is itself invariant, every separation falls into one of the three classes listed below.

Type	Condition	Physical meaning
Timelike	$\Delta s^2 < 0$	A causal signal slower than light can connect the events.
Null	$\Delta s^2 = 0$	A light ray can connect the events.
Spacelike	$\Delta s^2 > 0$	No causal signal can connect the events.

For a timelike separation there is a frame in which the two events happen at the same place, and for a spacelike separation there is a frame in which they happen at the same time. No Lorentz transformation moves a separation from one class into another.

The null condition $\Delta s^2 = 0$ defines the light cone. Every Lorentz transformation preserves this condition, so every inertial observer agrees on the speed of light. The separations inside the cone are timelike and those outside it are spacelike, and a Lorentz transformation moves a vector around *within* one of these regions but never carries it across the cone.

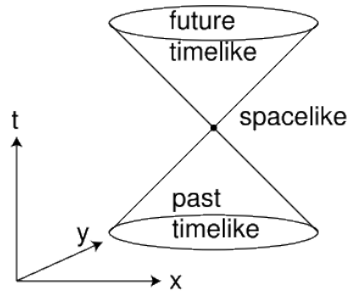


Figure 1: Light cone

3 Definition of a Lorentz transformation

> *A Lorentz transformation is a linear change of inertial coordinates that leaves the spacetime interval between any two events unchanged.*

The defining condition. A homogeneous linear coordinate change has the form $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$. If we demand that $x'^2 = x^2$ for every x, then we find

$$g_{\mu\nu} x'^{\mu} x'^{\nu} = g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} x^{\rho} x^{\sigma} \overset{!}{=} g_{\rho\sigma} x^{\rho} x^{\sigma},$$

which gives

$$\boxed{g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}} \qquad \Longleftrightarrow \qquad \boxed{\Lambda^{\mathsf{T}} g \Lambda = g.}$$

If we solve this condition directly, by treating the sixteen entries of Λ as unknowns, we find that six of them remain independent. We will meet those six parameters in §4, where three of them turn out to be ordinary rotation angles and the other three describe boosts.

This condition defines the Lorentz group $O(1,3)$. The letter O stands for the orthogonal group, which here means the group that preserves the Minkowski metric rather than the Euclidean one, and the pair $(1,3)$ records the signature of that metric.

The closest Euclidean analogue is $O(3)$, where the condition $R^{\mathsf{T}}R = \mathbf{1}$ preserves the ordinary length $\vec{x}^{\,2}$. Lorentz transformations may therefore be thought of as the "rotations" of Minkowski spacetime, and they mix time together with space.

The inverse matrix follows directly from the defining equation, and it is given by

$$\Lambda^{-1} = g^{-1}\Lambda^{\mathsf{T}}g = g\Lambda^{\mathsf{T}}g, \qquad \text{i.e.} \qquad (\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}.$$

Note that Λ^{-1} is not equal to Λ^{T}, that is the Lorentz matrix is generally *not* orthogonal in the Euclidean sense: what such a matrix preserves is g rather than $\mathbf{1}$.

Four components. Taking the determinant of both sides of $\Lambda^{\mathsf{T}}g\Lambda = g$ gives

$$(\det\Lambda)^2 \det g = \det g \quad \Longrightarrow \quad \det\Lambda = \pm 1,$$

A transformation is called *proper* when the determinant equals $+1$, in which case it preserves the orientation of spacetime, and it is called *improper* when the determinant equals -1. The proper Lorentz transformations form the group $SO(1,3)$, where the letter S records the restriction to $\det\Lambda = 1$.

Separately, the Lorentz condition $\Lambda^{\mathsf{T}}g\Lambda = g$ forces $|\Lambda^0{}_0| \geq 1$. If $\Lambda^0{}_0 \geq 1$, then future-directed timelike vectors stay future-directed, and the transformation is called *orthochronous*. If instead $\Lambda^0{}_0 \leq -1$, then the direction of time is reversed. The sign cannot change continuously, since it would have to cross the forbidden band $|\Lambda^0{}_0| < 1$. These two binary choices split $O(1,3)$ into the four disconnected components collected in the table below.

$\det\Lambda$	Sign of $\Lambda^0{}_0$	Interpretation
$+1$	$\geq +1$	Proper and orthochronous
-1	$\geq +1$	Improper but time-orientation preserving
$+1$	≤ -1	Proper but time reversing
-1	≤ -1	Improper and time reversing

The piece connected to the identity,

$$\det\Lambda = +1, \qquad \Lambda^0{}_0 \geq 1,$$

is called the *proper orthochronous Lorentz group* $SO^+(1,3)$, and it contains everything that can be reached from the identity by smoothly turning on rotation angles and boost velocities. Parity $P : (t,\vec{x}) \mapsto (t,-\vec{x})$ and time reversal $T : (t,\vec{x}) \mapsto (-t,\vec{x})$ live in the other components. They are discrete transformations, and they cannot be built up in this way. Relativistic theories always respect $SO^+(1,3)$, whereas P and T may or may not be symmetries of a given process.

Adding translations. If we also allow $x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu}$, then we obtain the *Poincaré group*, which is the full spacetime symmetry of relativistic quantum field theory in flat spacetime. A translation is not itself a Lorentz transformation, because it moves the origin, but it does preserve the separation between any two events. This group therefore has ten continuous parameters, which are the six Lorentz parameters together with the four translations.

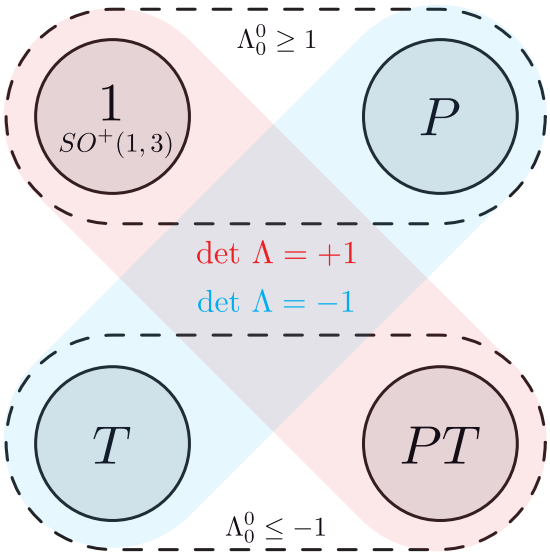


Figure 2: The four disconnected components of the Lorentz group. They are classified by the sign of $\Lambda^0{}_0$ (top versus bottom) and by $\det \Lambda$ (red versus blue). The components contain the identity 1, parity P, time reversal T, and their product PT, respectively.

4 Rotations and boosts

4.1 Spatial rotations

A rotation reorients the spatial axes without touching time. We simply embed a matrix $R \in SO(3)$ into the larger matrix

$$\Lambda_R = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix},$$

where the property $R^\mathsf{T} R = \mathbf{1}$ is what guarantees that $\Lambda_R^\mathsf{T} g \Lambda_R = g$. For example, the rotation around the x axis is given by

$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & -\sin\theta_x \\ 0 & 0 & \sin\theta_x & \cos\theta_x \end{pmatrix}.$$

Here θ_x is an ordinary angle, and it lies in the range $0 \leq \theta_i \leq 2\pi$.

4.2 Boosts

In a similar manner to the rotations we can define the boosts, which reorient a spatial axis together with time. For example, the boost in the x direction can be written as

$$B_x(\beta_x) = \begin{pmatrix} \cosh\beta_x & -\sinh\beta_x & 0 & 0 \\ -\sinh\beta_x & \cosh\beta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here we have defined a rapidity parameter β_i, which ranges over $-\infty < \beta_i < +\infty$. The rapidity β_x can be related to something more useful for us, namely the velocity v along the x axis,

$$\cosh\beta_x = \frac{1}{\sqrt{1-v^2}}, \quad \sinh\beta_x = \frac{v}{\sqrt{1-v^2}}$$

Rotation angles add when the rotations lie in a common plane, and rapidities add when the boosts lie along a common line, but velocities add in neither case. Rotation matrices and boost matrices along different directions do not commute.

Note that different sources use different symbols for the rapidity parameter.

5 Four-vectors and invariants

Strictly mathematically speaking, a four-vector is a set of four components that is *defined by a transformation law*,

$$\underbrace{V'^{\mu} = \Lambda^{\mu}{}_{\nu} V^{\nu}}_{\text{contravariant}}, \qquad \underbrace{V'_{\mu} = (\Lambda^{-1})^{\nu}{}_{\mu} V_{\nu} = \Lambda_{\mu}{}^{\nu} V_{\nu}}_{\text{covariant}}.$$

These two laws are what the index positions of the previous section actually mean. An object that transforms by the first law is called a *contravariant* vector, or simply a vector, and it carries its index upstairs. An object that transforms by the second law is called a *covariant* vector, or a *covector*, and it carries its index downstairs.

> The quick intuition behind the two names: a contravariant vector is an arrow, such as the displacement Δx^{μ}, so measuring it in smaller units makes its components larger, whereas a covariant vector is a gradient, such as ∂_{μ}, which is counted per unit of length, so the very same change of units makes its components smaller. The two objects therefore respond to a change of frame in opposite ways, which is what the prefixes "contra" and "co" are depict (vectors "contra"-dict varying of the coordinate basis, while covectors "co"-work with the basis).

A contravariant vector V^{μ} is an element of a vector space V, which here is Minkowski spacetime itself. A covariant vector V_{μ} instead lives in the *dual space* V^*, whose elements are the linear functionals that take a vector out of V and return a number. In this language x^{μ} gives the coordinates of a point in spacetime, while x_{μ} is the corresponding measuring stick that extracts projections (i.e. dot products) in a way consistent with the geometry set by the metric. Objects of the second kind are also called *one-forms*, precisely because they act on contravariant vectors and produce scalars.

Before contracting anything, it is worth seeing what happens if we do not. Taking two four-vectors V^{μ} and W^{ν} and simply multiplying them gives an object with two free indices, which is called a rank-two tensor, and it transforms with one copy of Λ for each index,

$$V'^{\mu} W'^{\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} V^{\alpha} W^{\beta}.$$

Its components certainly do change, and there is no reason at all for them to look the same to every observer, so this object is not Lorentz invariant.

What the metric supplies is the recipe for pairing an upper index with a lower one and collapsing the whole object into a single number. An upper index transforms with Λ and a lower index

transforms with Λ^{-1}, and that is precisely why the two cancel when we contract them, so that once no free index remains the result is the same in every inertial frame,

$$V'_\mu W'^\mu = (\Lambda^{-1})^\rho{}_\mu \Lambda^\mu{}_\sigma V_\rho W^\sigma = \delta^\rho{}_\sigma V_\rho W^\sigma = V_\rho W^\rho .$$

Written out in components this contraction is

$$V \cdot W \equiv g_{\mu\nu} V^\mu W^\nu = -V^0 W^0 + \vec{V} \cdot \vec{W},$$

and although each of V^0, V^i, W^0 and W^i depends on the frame, the full contraction does not.

Example 1: the mass shell. The four-momentum of a particle is $p^\mu = (E, \vec{p})$, so $p_\mu = (-E, \vec{p})$ and its contraction with itself is

$$p^2 = p_\mu p^\mu = -E^2 + \vec{p}^{\,2} = -m^2 \qquad \Longleftrightarrow \qquad \boxed{E^2 = \vec{p}^{\,2} + m^2.}$$

Every observer computes the same value of p^2, and that common value is what *defines* the mass. Energy on its own is not invariant, because under a boost E and $\vec{p}$ mix in exactly the same way as t and $\vec{x}$ do in §4.2, both being components of a single four-vector. A massive particle has a rest frame in which $p^\mu = (m, \vec{0})$. A massless particle has $p^2 = 0$, so that $E = |\vec{p}|$, and for it no rest frame exists. Sums of four-momenta contract in exactly the same way, so combinations such as $(p_1 + p_2)^2$ are invariant as well, and that is how the mass of a system of several particles is defined.

Example 2: the d'Alembertian. The derivative $\partial_\mu \equiv \partial/\partial x^\mu = (\partial_t, \vec{\nabla})$ is a covector, as the chain rule shows by giving $\partial'_\mu = (\Lambda^{-1})^\nu{}_\mu \partial_\nu$. Its contraction with itself is therefore a scalar,

$$\Box \equiv \partial_\mu \partial^\mu = -\partial_t^2 + \nabla^2 \qquad \Longrightarrow \qquad \boxed{(\Box - m^2)\phi = 0}$$

It follows that the Klein–Gordon equation takes the same form in every frame.

5.1 What is and is not invariant

Three pieces of vocabulary are worth fixing here, because they are easy to confuse. A quantity is *Lorentz invariant* if it takes the same value in every inertial frame related by a Lorentz transformation. The simplest such quantities are the *Lorentz scalars*, which are built by contracting indices until none is left free, as with p^2 or with $V_\mu W^\mu$ above. An object that still carries a free index, such as V^μ itself, is not a scalar, since its components change from one frame to the next. It is called *Lorentz covariant* instead, which means that all observers agree that it is the same vector written out in different coordinates. Finally, an object is not invariant at all when it fails to transform by the Lorentz rules in the first place, so that its very definition depends on which frame we happened to choose (e.g. the singular component of the vector).

Counting free indices is therefore a quick and surprisingly powerful check on any expression. If no free index is left, then the result had better be a scalar. If a single free index remains, then the expression should transform as a vector or as a covector carrying that index. And if the two sides of an equation carry different free indices, then the equation is already wrong.

6 How fields transform

It is worth noting that a *scalar field* is not a field with the same value everywhere, because it may vary arbitrarily through spacetime. The word "scalar" means that at a fixed physical event

its value has no components for a Lorentz transformation to mix. Temperature in a room is the familiar analogy, because it varies from place to place and is nevertheless a scalar under rotations.

Under a Lorentz transformation the scalar field stays the same, so that $\phi(x) \to \phi(x)$. Sometimes people write instead that the coordinate changes, as in

$$\phi'(x') = \phi(x) = \phi(\Lambda^{-1}x'), \qquad x' = \Lambda x,$$

This form is a little confusing, since it makes it look as though the field itself changes under the transformation. It does not, and the shift in the coordinate does not tell us anything useful about the physics. Consider the example of temperature once more. If we move the coordinate grid, then the labelling of the points changes, but the temperature at each point stays the same. A scalar field is therefore just a number attached to each point of spacetime, no matter how we choose to relabel those points.

A vector field changes in the way that a vector does,

$$A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu}.$$

One example of such a field is the electric four-current.

A tensor field changes in the way that a tensor does,

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} T^{\alpha\beta}.$$

One example of such a field is the stress-energy tensor.

7 Infinitesimal transformations and generators

> Here I will be brief, since this part is not particularly relevant to the current part of the course and the Lorentz group generators and their representation will become much more important later in the course. For more details read Srednicki Chapter 2.

Every element of $SO^+(1,3)$ can be reached from the identity along a continuous path, so it is enough to understand the transformations that sit infinitesimally close to it. For such a transformation we write

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu} + O(\delta\omega^2).$$

Substituting this into $\Lambda^{\mathsf{T}} g \Lambda = g$ and keeping only the terms of first order gives

$$\delta\omega_{\mu\nu} + \delta\omega_{\nu\mu} = 0, \qquad \text{where} \qquad \delta\omega_{\mu\nu} \equiv g_{\mu\lambda}\, \delta\omega^{\lambda}{}_{\nu}.$$

An antisymmetric 4×4 array has $\frac{4\cdot 3}{2} = 6$ independent components, so a Lorentz transformation near the identity is fixed by six parameters. The three purely spatial components $\delta\omega_{ij}$ parametrise the rotations, and the three mixed components $\delta\omega_{0i}$ parametrise the boosts.

7.1 Exponentiating the generators

Nothing is lost by working infinitesimally, because a finite transformation is recovered by compounding N small ones and letting $N \to \infty$. Setting $\delta\omega = \omega/N$,

$$\Lambda = \lim_{N\to\infty} \Big(\mathbf{1} + \frac{\omega}{N}\Big)^N = \boxed{\exp\omega}$$

Splitting ω into its rotation and boost parts introduces the six *generators*, which are the three rotation generators J_i coming from the ω_{ij} and the three boost generators K_i coming from the ω_{0i}. The rotations and boosts are then written as exponentials of their generators as

$$\boxed{R_i(\theta) = e^{-i\theta J_i}, \qquad B_i(\beta) = e^{i\beta K_i},}$$

and a general proper orthochronous transformation is

$$\Lambda(\vec{\theta}, \vec{\beta}) = \exp(-i\vec{\theta} \cdot \vec{J} + i\vec{\beta} \cdot \vec{K}).$$

The explicit matrices are collected in the reference sheet at the back. Expanding $e^{-i\theta J_z}$ returns the block built from $\cos\theta$ and $\sin\theta$, and expanding $e^{i\beta K_x}$ returns the block built from $\cosh\beta$ and $\sinh\beta$.

These six matrices close under commutation, and they satisfy

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad [J_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k.$$

8 Preview: Dimensional Analysis

In quantum field theory we often work in natural units, in which $\hbar = c = 1$.

One major consequence of this choice is that many of the units we are used to keeping apart become the same,

$$[m] = [p] = [E], \qquad [1/L] = [1/T] = [E].$$

This choice makes life much simpler when we are doing quantum field theory calculations. At the end of a calculation, on the other hand, we usually need to return to the usual SI units. Fortunately this is easier than it may seem, because in order to translate a result expressed in natural units into SI units we only have to multiply it by enough powers of $\hbar$ and c to obtain the dimensions we want. We sometimes also need k_B, if a temperature is involved.

As an example, consider the Casimir force in $1+1$ dimensions between two plates that are separated by a distance a. In natural units it is given by

$$F = -\frac{\pi}{24a^2}.$$

In order to restore the units, let us write it instead as

$$F = -\hbar^\alpha c^\beta \frac{\pi}{24a^2},$$

where α and β must be determined from dimensional analysis.

On the left-hand side the dimensions are

$$[F] = \frac{[M][L]}{[T]^2}.$$

On the right-hand side they are

$$\left[\hbar^\alpha c^\beta \frac{1}{a^2}\right] = \frac{1}{[L]^2} \underbrace{\left(\frac{[M][L]^2}{[T]}\right)^\alpha}_{[\hbar]^\alpha} \underbrace{\left(\frac{[L]}{[T]}\right)^\beta}_{[c]^\beta}.$$

Matching the powers of M, L, and T on the two sides then gives

$$\alpha = \beta = 1.$$

Therefore the Casimir force in SI units is given by

$$F = -\frac{\pi \hbar c}{24 a^2}.$$

A similar procedure can be carried out by writing $\hbar c$ in units of GeV fm and c in units of m/s. In this way we can translate an energy scale E given in GeV into other SI units, such as a mass, an inverse length, or an inverse time.

9 Reference sheet: explicit matrices

9.1 Spatial rotations

The three elementary rotations are the following.

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}, \; R_y(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}, \; R_z(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

A general rotation is built from these as $R = R_z(\phi) R_y(\vartheta) R_x(\psi)$. The Euler angles are named ϕ, ϑ, ψ here so that γ is never anything but the Lorentz factor.

9.2 Lorentz boosts

The three elementary boosts are the following.

$$B_x(\beta) = \begin{pmatrix} \cosh\beta & -\sinh\beta & 0 & 0 \\ -\sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_y(\beta) = \begin{pmatrix} \cosh\beta & 0 & -\sinh\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\beta & 0 & \cosh\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_z(\beta) = \begin{pmatrix} \cosh\beta & 0 & 0 & -\sinh\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\beta & 0 & 0 & \cosh\beta \end{pmatrix}.$$

For a boost in a general direction $\vec{v} = (v_x, v_y, v_z)$, with $v^2 = v_x^2 + v_y^2 + v_z^2$ and $\gamma = (1 - v^2)^{-1/2}$ in units where $c = 1$, the matrix is

$$B(\vec{v}) = \begin{pmatrix} \gamma & -\gamma v_x & -\gamma v_y & -\gamma v_z \\ -\gamma v_x & 1 + \dfrac{\gamma - 1}{v^2} v_x^2 & \dfrac{\gamma - 1}{v^2} v_x v_y & \dfrac{\gamma - 1}{v^2} v_x v_z \\ -\gamma v_y & \dfrac{\gamma - 1}{v^2} v_y v_x & 1 + \dfrac{\gamma - 1}{v^2} v_y^2 & \dfrac{\gamma - 1}{v^2} v_y v_z \\ -\gamma v_z & \dfrac{\gamma - 1}{v^2} v_z v_x & \dfrac{\gamma - 1}{v^2} v_z v_y & 1 + \dfrac{\gamma - 1}{v^2} v_z^2 \end{pmatrix}.$$

9.3 Parity and time reversal

In matrix form these two discrete transformations are the following.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

9.4 Generators

The generators are defined by $R_i(\theta) = e^{-i\theta J_i}$ and $B_i(\beta) = e^{i\beta K_i}$. The three rotation generators are the following.

$$J_x = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad J_y = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad J_z = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The three boost generators are the following.

$$K_x = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_y = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_z = i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$