Section 2: Contour Integration and Path Integrals

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Schwartz, Ch. 3, following O. Barrera's 2025 QFT1 section notes

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1 Why do we need contours?

Sooner or later in the quantum field theory course one runs into an integral of the form

$$I(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \, \frac{e^{-\mathrm{i}k^0 t}}{(k^0)^2 - \omega_k^2 + \mathrm{i}\epsilon}, \tag{1}$$

where t is a real time, ω_k is a fixed positive number, and ϵ is a positive number that we send to zero at the end. The integration variable k^0 runs over the whole real line, so this is a perfectly ordinary real integral with a complex-valued integrand (usually appearing from a Fourier transform), and the answer is a single complex number for each t.

The trouble in this integral sits in the denominator. At $k^0 = \pm\omega_k$ the quantity $(k^0)^2 - \omega_k^2$ vanishes, and were it not for the $\mathrm{i}\epsilon$ the integrand would be infinite at those two points. With ϵ small but nonzero the integrand is finite everywhere on the real axis, but it grows greatly in two narrow spikes, and the value of $I(t)$ is controlled almost entirely by what happens there. In order to solve this integral, substitution does not help, partial fractions do not help, and no table of real integrals has this one in it. Furthermore, the oscillating factor $e^{-\mathrm{i}k^0 t}$ has modulus one all along the real axis, so it never helps the integral converge and it never lets us throw away any part of the range either.

The trick for an integral of this kind is to stop thinking of k^0 as a real number and let it move into the complex plane. The integrand becomes a function of a complex variable with two bad points sitting just off the real axis, the real line becomes one particular path through that plane, and it turns out that we are allowed to close that path off with a large arc and then slide the whole thing around like a rubber band, so long as we never drag it across a bad point. Choosing which way to close reduces the integral to contributions from those two points alone and, as it happens, from only one of them at a time. So what was hopeless becomes a few lines of algebra. The rest of Part I builds the machinery that makes this legitimate, and we come back and collect the answer in Eq. (30).

2 A very short tour of the complex plane

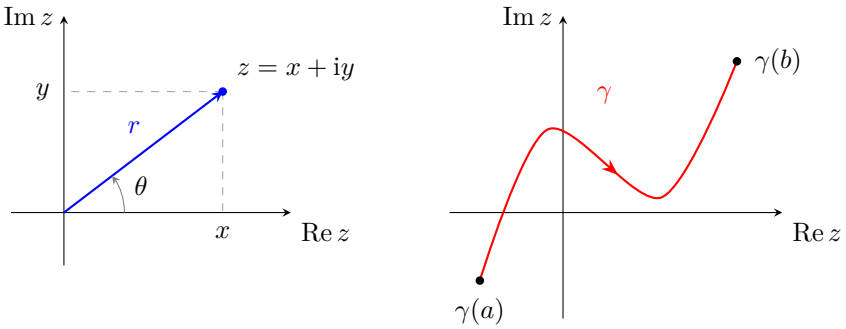
A complex number $z = x + \mathrm{i}y$ is a pair of real numbers, and a way to picture it is as a point of the plane with horizontal coordinate $x = \mathrm{Re}\, z$ and vertical coordinate $y = \mathrm{Im}\, z$. That plane, with those two axes, is called *the complex plane*.

Addition of complex numbers is then just addition of the corresponding arrows, tip to tail. Multiplication is much more interesting. Writing the same number in polar form,

$$z = r\, e^{\mathrm{i}\theta}, \qquad r = |z| = \sqrt{x^2 + y^2}, \qquad \theta = \arg z, \tag{2}$$

we can read multiplication off immediately: $(r_1 e^{\mathrm{i}\theta_1})(r_2 e^{\mathrm{i}\theta_2}) = r_1 r_2\, e^{\mathrm{i}(\theta_1+\theta_2)}$, so multiplying by a complex number stretches by its modulus and rotates by its argument. So we can think of every complex number as a stretch-and-turn instruction.

A *path*, or *contour*, is then just a curve drawn in that plane, with an arrow on it telling us which way we are walking along it.



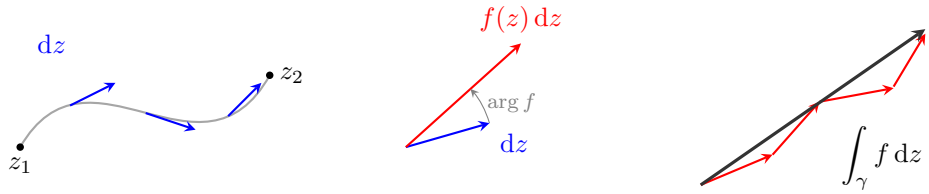
(a) a point, $z = r\,e^{i\theta}$ (b) a contour, drawn in the same plane

Figure 1: Left: a complex number is a point of the plane, equivalently an arrow with length r and direction θ. Right: a contour is a curve drawn in that plane and given a direction of travel.

3 What a contour integral is

3.1 The picture

When we first meet integrals we picture them as areas under curves. That works beautifully on the real line, but in the complex plane there is no natural "under", and there is no area to speak of. So here is the replacement. The value $f(z)$ is itself a complex number, which, as we just saw, is a stretch-and-turn instruction. As we walk along the contour, each tiny step we take is also a complex number, call it $\mathrm{d}z$: a little arrow pointing in the direction we happen to be walking at that instant. The recipe is to take each step, hand it to the function sitting at that point, let f stretch it by $|f(z)|$ and rotate it by $\arg f(z)$, and then lay all of the resulting arrows end to end. Their vector sum is the single net arrow from the beginning of the chain to its end, and this net arrow represents the contour integral.



(a) steps along the path (b) turn and stretch (c) laid end to end

Figure 2: Reading a contour integral. Each step $\mathrm{d}z$ along the path is multiplied by the local value of f, which turns it and changes its length, and the resulting arrows are laid end to end. The integral is then the single arrow from the start of that chain to its end.

3.2 Saying the same thing with symbols

The path is a rule assigning, to each moment t of some time interval, the point of the plane where we are standing at that moment:

$$\gamma : [a, b] \to \mathbb{C}, \qquad t \longmapsto \gamma(t), \tag{3}$$

which we assume continuous and with a piecewise continuous derivative, so that the curve has no jumps and has a well defined direction almost everywhere.

With that said, the picture of the previous subsection becomes

$$\boxed{\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t, \qquad \mathrm{d}z = \gamma'(t)\,\mathrm{d}t.} \tag{4}$$

The right-hand side is an ordinary integral over the real interval $[a, b]$ of a complex-valued function of t, which we handle by splitting it into real and imaginary parts and integrating each with the calculus we already have. The one genuinely new ingredient is a complex number $\mathrm{d}z = \gamma'(t)\,\mathrm{d}t$, which is precisely the little arrow of Figure 2 keeping track of the direction we are moving at each instant.

The simplest possible case. Take $f(z) = 1$, so that nothing gets stretched or turned at all. Then

$$\int_\gamma \mathrm{d}z = \int_a^b \gamma'(t)\,\mathrm{d}t = \gamma(b) - \gamma(a). \tag{5}$$

The answer depends only on where the path starts and where it ends, and not at all on the route we took between them: two wildly different curves with the same endpoints give the same number, and any closed curve gives zero. This agrees with the picture, since laying the untouched steps end to end simply retraces the path.

Orientation. Reversing the path means replacing $\gamma(t)$ by $\tilde{\gamma}(t) = \gamma(a + b - t)$, which walks over the same set of points in the opposite direction. Then $\tilde{\gamma}'(t) = -\gamma'(a + b - t)$, and changing variables in Eq. (4) gives

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z. \tag{6}$$

For a closer contour we fix that *positively oriented* means counterclockwise.

Remark: dz is not arc length. Alongside $\mathrm{d}z = \gamma'(t)\,\mathrm{d}t$ there is the arc length element $\mathrm{d}s = |\gamma'(t)|\,\mathrm{d}t$, a positive real number measuring how far we have moved, with no memory of the direction. Integrating $\mathrm{d}s$ around a closed curve gives its total length, which is certainly not zero, whereas integrating $\mathrm{d}z$ around the same curve gives zero by Eq. (5). The two are not interchangeable. Arc length is important later, when we want to bound an integral through

$$\Big|\int_\gamma f\,\mathrm{d}z\Big| \le \int_\gamma |f|\,\mathrm{d}s \le \big(\max_\gamma |f|\big) \times (\text{length of } \gamma), \tag{7}$$

which just says that a chain of arrows cannot reach further than the sum of their lengths.

4 Antiderivatives

On the real line, essentially the only thing we ever do with an integral is find an antiderivative and evaluate it at the endpoints. If $f = F'$ then $\int_a^b f = F(b) - F(a)$, and the entire apparatus of first-year calculus is a collection of tricks for guessing F. The first question to ask about our new integral is therefore whether that trick survives, and the reassuring answer is that it does, almost word for word. *"Almost"*.

Suppose then that $f = F'$ for some F defined on a region containing the contour, where the derivative $F'(z)$ is computed as the usual limit of a difference quotient, with one important

caveat: the increment in the denominator of the quotient is now a complex number, and *we demand the same limit no matter which direction it comes in from.* Let γ run from $z_1 = \gamma(a)$ to $z_2 = \gamma(b)$. Then the chain rule applies along the path,

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t)) = F'(\gamma(t))\,\gamma'(t) = f(\gamma(t))\,\gamma'(t). \tag{8}$$

The right-hand side of Eq. (8) is the integrand of Eq. (4), so

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t))\,\mathrm{d}t = F(z_2) - F(z_1). \tag{9}$$

So the old trick is intact: the value depends only on the endpoints and not at all on the shape of the curve. A closed contour has $z_2 = z_1$, and so

$$\boxed{\oint_\gamma f(z)\,\mathrm{d}z = 0 \quad \text{whenever } f = F' \text{ for some } F \text{ on a region containing } \gamma.} \tag{10}$$

A concrete check. Let C_R be the upper semicircle of radius R centered at the origin, oriented counterclockwise, and take $f(z) = z^2$, whose antiderivative $z^3/3$ is single valued everywhere.

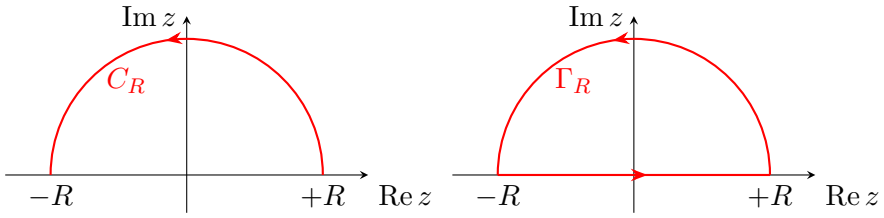


Figure 3: The upper semicircle C_R, oriented counterclockwise.

Parametrising by $\gamma(\theta) = Re^{\mathrm{i}\theta}$ with $\theta \in [0, \pi]$ gives $\mathrm{d}z = \mathrm{i}Re^{\mathrm{i}\theta}\mathrm{d}\theta$, so

$$\int_{C_R} z^2\,\mathrm{d}z = \int_0^\pi (R^2 e^{2\mathrm{i}\theta})(\mathrm{i}Re^{\mathrm{i}\theta})\mathrm{d}\theta = \mathrm{i}R^3 \int_0^\pi e^{3\mathrm{i}\theta}\,\mathrm{d}\theta = \mathrm{i}R^3 \left[\frac{e^{3\mathrm{i}\theta}}{3\mathrm{i}}\right]_0^\pi = -\tfrac{2}{3}R^3.$$

The answer depends on R, which is to say on the contour we chose, as it must, since the semicircle on its own is not closed. Now close it by adding the diameter along the real axis from $-R$ to $+R$, giving a closed loop Γ_R. Along the diameter the variable is real, $z = x$, so

$$\int_{-R}^{R} z^2\,\mathrm{d}z = \int_{-R}^{R} x^2\,\mathrm{d}x = \tfrac{2}{3}R^3,$$

and adding the two pieces gives

$$\oint_{\Gamma_R} z^2\,\mathrm{d}z = \tfrac{2}{3}R^3 + (-\tfrac{2}{3}R^3) = 0,$$

in agreement with Eq. (10), and for every radius R.

So where is the "almost"? Two separate things can go wrong. There may simply be no F at all. Or there may be an F which exists perfectly well but fails to come back to its own value once we have walked all the way around the loop, in which case Eq. (9) is still true and Eq. (10) is not, because $F(z_2)$ and $F(z_1)$ are two different numbers even though z_1 and z_2 are the same point. As it happens, the most useful function in this course is of the second kind.

5 The important exception: $1/z$

Does every closed contour integral vanish, then? The antiderivative argument covers an enormous amount of ground, so rather than cataloguing everything it works for, it is much quicker to go hunting for a place where it fails. Take the simplest function with a bad point, $f(z) = 1/z$, and integrate it around the circle $|z| = R$ traversed once counterclockwise. Parametrising by $z = Re^{\mathrm{i}\theta}$ with $\theta \in [0, 2\pi]$ and $\mathrm{d}z = \mathrm{i}Re^{\mathrm{i}\theta}\mathrm{d}\theta$,

$$\oint_{|z|=R} \frac{\mathrm{d}z}{z} = \int_0^{2\pi} \frac{\mathrm{i}Re^{\mathrm{i}\theta}}{Re^{\mathrm{i}\theta}}\,\mathrm{d}\theta = \mathrm{i}\int_0^{2\pi} \mathrm{d}\theta = 2\pi\mathrm{i}.$$

Not zero. Note also that the radius cancelled between numerator and denominator before the integration even started, so the answer is the same $2\pi\mathrm{i}$ for a circle of any radius, however large or small.

Before drawing any conclusions, let us check whether the culprit is really the singularity. Run the same parametrisation for the whole family $f(z) = 1/z^n$ with $n = 1, 2, 3, \ldots$, every one of which blows up at the origin, and most of them more violently than $1/z$ does,

$$\oint_{|z|=R} \frac{\mathrm{d}z}{z^n} = \int_0^{2\pi} \frac{\mathrm{i}Re^{\mathrm{i}\theta}}{R^n e^{\mathrm{i}n\theta}}\,\mathrm{d}\theta = \mathrm{i}R^{\,1-n}\int_0^{2\pi} e^{\mathrm{i}(1-n)\theta}\,\mathrm{d}\theta.$$

The remaining integral runs over a whole number of periods of a complex exponential, so it vanishes for every $n \neq 1$. Only when $n = 1$ does the exponent collapse to zero, leaving the constant integrand 1 and the value 2π. Then

$$\boxed{\oint_{|z|=R} \frac{\mathrm{d}z}{z^n} = 0 \quad (n \in \mathbb{Z},\ n \neq 1), \qquad \oint_{|z|=R} \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i}.} \tag{11}$$

The vanishing cases hold even though the contour encircles a point where the function is infinite. So a singularity by itself is not what makes a closed contour integral nonzero.

What distinguishes $n = 1$ is, once again, the antiderivative. For $n \neq 1$ we have

$$\frac{1}{z^n} = \frac{\mathrm{d}}{\mathrm{d}z}\frac{z^{1-n}}{1-n},$$

and z^{1-n} is single valued away from the origin. Walk once around the circle and it returns to the value it started from, the endpoints in Eq. (9) cancel, and we get zero. For $n = 1$ the antiderivative is instead

$$\frac{\mathrm{d}}{\mathrm{d}z}\log z = \frac{1}{z},$$

and here $\log z$ is not single valued. One turn around the origin does not bring it back to where it started, so the hypothesis behind Eq. (10) is violated. The $2\pi\mathrm{i}$ we computed is precisely the amount by which the logarithm fails to close up.

5.1 Complex logarithms, branches and cuts

Every nonzero complex number can be written in the polar form Eq. (2), and the *argument* θ in it is defined only up to adding multiples of 2π, since turning once around the origin does not change z at all but does change the angle we used to describe it by 2π. That ambiguity is harmless for z itself, but not for its logarithm.

A logarithm of z is any w satisfying $e^w = z$. Writing $w = a + \mathrm{i}b$ with a, b real, the equation $e^a e^{\mathrm{i}b} = re^{\mathrm{i}\theta}$ matches magnitudes and angles to give $a = \ln r$ and $b = \theta + 2\pi k$ for any integer k, so the logarithm is *multivalued*,

$$\log z = \ln r + \mathrm{i}(\theta + 2\pi k), \qquad k \in \mathbb{Z}. \tag{12}$$

Picture a spiral staircase built over the plane. The function "wants" to be $\ln r + \mathrm{i}\theta$, but walking once around the origin sends $\theta \mapsto \theta + 2\pi$ and lifts us to the next floor, a height $2\pi\mathrm{i}$ above the one we started on.

Since we would like a function, and a function is supposed to return one number, we make a choice. A *branch* of the logarithm is a choice of one floor of that staircase, and it is made by fixing a continuous range for θ, say $\theta \in (-\pi, \pi]$. But such a choice cannot be continuous everywhere: somewhere the angle has to jump, and the locus where it jumps is a line running from the origin out to infinity, which we remove from the domain and call the *branch cut*. It may be placed anywhere, provided it runs from the singular point out to infinity, and different textbooks place it differently. The function is well behaved on either side of the cut and discontinuous across it.



Figure 4: Left: as we go round and round one circle in the z plane, the logarithm climbs a staircase, rising by $2\pi\mathrm{i}$ each turn. The three marked points sit above the same z and carry the three values $\ln r + \mathrm{i}\theta$, $\ln r + \mathrm{i}(\theta + 2\pi)$ and $\ln r + \mathrm{i}(\theta + 4\pi)$. Right: a branch keeps one floor by cutting the plane, so that no path can complete a turn. The two sides of the cut differ by 2π in $\operatorname{Arg} z$.

The standard choice is the *principal branch*,

$$\operatorname{Arg} z \in (-\pi, \pi], \qquad \operatorname{Log} z \equiv \ln|z| + \mathrm{i}\operatorname{Arg} z, \tag{13}$$

whose cut runs along the negative real axis. We write Log for the principal branch and log for the multivalued object of Eq. (12). On the positive real axis the angle is $\theta = 0$, so $\operatorname{Log} x = \ln x$ and the ordinary real logarithm is recovered with no jump, which is why this branch is the standard choice.

Approaching the point -1 from above the cut gives angle $+\pi$ and the value $\operatorname{Log}(-1) = \mathrm{i}\pi$, while approaching from below gives angle $-\pi$ and the value $-\mathrm{i}\pi$. The jump across the cut is therefore $\mathrm{i}\pi - (-\mathrm{i}\pi) = 2\pi\mathrm{i}$, the height of one step of the staircase, and it is the same $2\pi\mathrm{i}$ that came out of $\oint \mathrm{d}z/z$ in Eq. (11), for the same reason. A contour encircling the origin must cross the cut once, and in crossing it the antiderivative climbs one floor instead of returning to its initial value.

6 Holomorphic functions and Cauchy's theorem

So, if f has a single-valued antiderivative on a region containing the contour, closed loops give zero by Eq. (10); if it has none, as happens for $1/z$ on any region enclosing the origin, the loop

can give something non-zero. That is an uncomfortable place to stop, because in practice we are handed f and not F. What we want is a condition on f itself, checkable by looking at f, which guarantees that a suitable F exists.

A few words we will need. Since the statements below are careful about where things hold, let us fix the vocabulary. An *open set* $U \subseteq \mathbb{C}$ is a set in which every point has a little disc around it that also lies in U; equivalently, a set with no boundary points of its own, so that from anywhere inside it one can still move a little in any direction. A *neighbourhood* of z is such a little disc around z. A *region* or *domain* is an open set that is connected, meaning it is all one piece. A *simple closed contour* is a closed path that does not cross itself, so that it has an inside and an outside, and a region is *simply connected* if every closed contour in it can be shrunk to a point without leaving the region, which is just a way of saying that the region has no holes punched in it. The punctured plane $\mathbb{C} \setminus \{0\}$ is the standard example of a region that is *not* simply connected.

Complex differentiability. The function f is differentiable at $z \in \mathbb{C}$ if the limit

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{14}$$

exists, where h is now a complex increment. The point $z+h$ may approach z from any direction in the plane, along the real axis, along the imaginary axis, or along a spiral, and every one of those approaches is required to return the same number $f'(z)$. Real differentiability asks for agreement between two approaches, from the left and from the right; complex differentiability asks for agreement between a continuum of them.

The standard illustration is $f(z) = |z|^2$, which is a smooth function of (x, y) and yet complex-differentiable only at the origin. Write $|z|^2 = z\bar{z}$, so that

$$\frac{|z+h|^2 - |z|^2}{h} = \frac{(z+h)(\bar{z}+\bar{h}) - z\bar{z}}{h} = z\,\frac{\bar{h}}{h} + \bar{z} + \bar{h}.$$

The last two terms are harmless, since $\bar{z}$ does not involve h at all and $\bar{h} \to 0$. Everything hinges on $\bar{h}/h$, and setting $h = |h|e^{\mathrm{i}\theta}$ gives

$$\frac{\bar{h}}{h} = \frac{|h|e^{-\mathrm{i}\theta}}{|h|e^{\mathrm{i}\theta}} = e^{-2\mathrm{i}\theta},$$

which does not settle down as $|h| \to 0$, however small h is made, but retains a memory of the direction θ of approach. Approaching along the real axis gives $+1$ and along the imaginary axis gives -1. The limit therefore exists only when the coefficient multiplying $e^{-2\mathrm{i}\theta}$ vanishes, namely at $z = 0$.

Holomorphic. The function f is *holomorphic* on an open set U if it is complex differentiable at every point of U. We insist that U be open precisely because the definition needs room: to approach z from every direction we need a whole neighbourhood of z inside U, and a boundary point would not supply one.

Direction-independence can be repackaged as a pair of partial differential equations. Write $f = u + \mathrm{i}v$ with u and v real functions of x and y. Taking h real in Eq. (14) gives $f' = \partial_x u + \mathrm{i}\partial_x v$, while taking h purely imaginary gives $f' = -\mathrm{i}\partial_y u + \partial_y v$. Demanding that these two agree, and matching real and imaginary parts, produces the *Cauchy–Riemann equations*

$$\boxed{\partial_x u = \partial_y v, \qquad \partial_y u = -\partial_x v.} \tag{15}$$

The familiar functions e^z, z^n, $\sin z$ and $\cos z$ satisfy them everywhere, and any rational function satisfies them away from the zeros of its denominator, which covers essentially every integrand in this course.

A holomorphic function is automatically infinitely differentiable, and more than that, it is *analytic*, meaning that near any point z_0 of its domain it equals a convergent power series $\sum_{n\geq 0} a_n (z - z_0)^n$. Nothing remotely like this holds on the real line, where a function can be differentiable once and not twice, or infinitely differentiable and still not equal to its own Taylor series. In the complex plane a single derivative, existing in every direction, forces the entire rigid structure into place.

Cauchy's theorem.

$$\boxed{f \text{ holomorphic on and inside a closed contour } C \quad \Longrightarrow \quad \oint_C f(z)\,\mathrm{d}z = 0.} \tag{16}$$

It can be proved directly from the Cauchy–Riemann equations Eq. (15), as we sketch below. And once we have it, the antiderivative comes back to us for free: the integral $F(z) = \int_{z_0}^{z} f(w)\,\mathrm{d}w$ no longer depends on which path we take from z_0 to z, because the difference of two such paths is a closed contour and gives zero, so it genuinely defines a function, and that function satisfies $F' = f$.

It also explains the behaviour of $1/z$ found in Eq. (11). The function $1/z$ is holomorphic on the punctured plane, so there is nothing wrong with it anywhere along the circle itself. But Eq. (16) demands holomorphy on *and inside* the contour, and the origin sits inside. One bad point in the enclosed region is enough to void the conclusion. Conversely, a closed contour that does not enclose the origin gives zero for $1/z$ as well, in complete agreement with the theorem.

6.1 Where the theorem comes from

The following justification uses Green's theorem from vector calculus, and a reader willing to take Eq. (16) on trust can skip it without losing the thread.

Write $z = x + \mathrm{i}y$, $\mathrm{d}z = \mathrm{d}x + \mathrm{i}\,\mathrm{d}y$ and $f = u + \mathrm{i}v$ with u, v real. Multiplying out,

$$(u + \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\,\mathrm{d}y) = \underbrace{(u\,\mathrm{d}x - v\,\mathrm{d}y)}_{\text{real part}} + \mathrm{i}\,\underbrace{(v\,\mathrm{d}x + u\,\mathrm{d}y)}_{\text{imaginary part}},$$

so that

$$\int_\gamma f\,\mathrm{d}z = \int_\gamma (u\,\mathrm{d}x - v\,\mathrm{d}y) + \mathrm{i}\int_\gamma (v\,\mathrm{d}x + u\,\mathrm{d}y).$$

Each piece is the circulation $\int_\gamma \mathbf{A}\cdot\mathrm{d}\mathbf{r}$ of a planar vector field, exactly as in electromagnetism, with

$$\mathbf{A}_1 = (u, -v), \qquad \mathbf{A}_2 = (v, u).$$

Green's theorem converts the circulation around a closed curve into the curl integrated over the region it bounds: for a region D with positively oriented boundary ∂D and a field $\mathbf{A} = (P, Q)$,

$$\oint_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\mathrm{d}A.$$

Applied to our two fields this gives

$$\oint \mathbf{A}_1 \cdot \mathrm{d}\mathbf{r} = \iint_D (-\partial_x v - \partial_y u)\mathrm{d}A, \qquad \oint \mathbf{A}_2 \cdot \mathrm{d}\mathbf{r} = \iint_D (\partial_x u - \partial_y v)\mathrm{d}A,$$

and if f is holomorphic on D then the Cauchy–Riemann equations Eq. (15) make both integrands vanish point by point, both circulations vanish, and $\oint_C f\,\mathrm{d}z = 0$. So holomorphy is an irrotational condition in disguise, and Cauchy's theorem is the statement that an irrotational field has no circulation. Where holomorphy fails the curl acquires a source, and the loop integral is no longer obliged to vanish.

7 Contour deformation and Cauchy's integral formula

If a closed loop gives zero whenever it encloses no singularity, then two loops enclosing the same singularities must give the same answer, and this is how Cauchy's theorem is actually used in practice. Let C_1 and C_2 be two contours with the same endpoints, and suppose C_1 can be slid continuously onto C_2 without any part of it ever crossing a point where f fails to be holomorphic. Follow C_1 forwards and then C_2 backwards; using Eq. (6), the result is a closed loop bounding exactly the region swept out by the deformation, and f is holomorphic throughout that region by assumption, so Eq. (16) applies to it,

$$\int_{C_1} f\,\mathrm{d}z - \int_{C_2} f\,\mathrm{d}z = \oint_{\partial D} f\,\mathrm{d}z = 0,$$

and therefore

$$\boxed{\int_{C_1} f\,\mathrm{d}z = \int_{C_2} f\,\mathrm{d}z.} \tag{17}$$

In other words, contours are made of rubber. A path may be pushed, stretched, inflated or collapsed at will and the integral does not notice, as long as no singularity is dragged across in the process. A singularity is a pin holding the rubber sheet down, and the only thing a contour remembers about itself is which pins it goes around; everything else about it, its shape, its length, is forgotten.

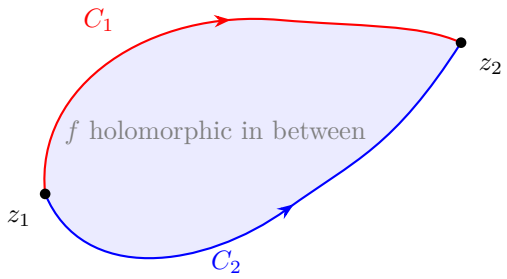


Figure 5: Deformation. The paths C_1 and C_2 share their endpoints, and the shaded region caught between them contains no singularity, so the two integrals agree. The point where f misbehaves lies outside, and no amount of pushing the path around is allowed to drag it across.

7.1 Cauchy's integral formula

Let f be holomorphic on a domain D, let C be a positively oriented simple closed contour lying in D, and let a be a point in the interior of C, as in Figure 6.

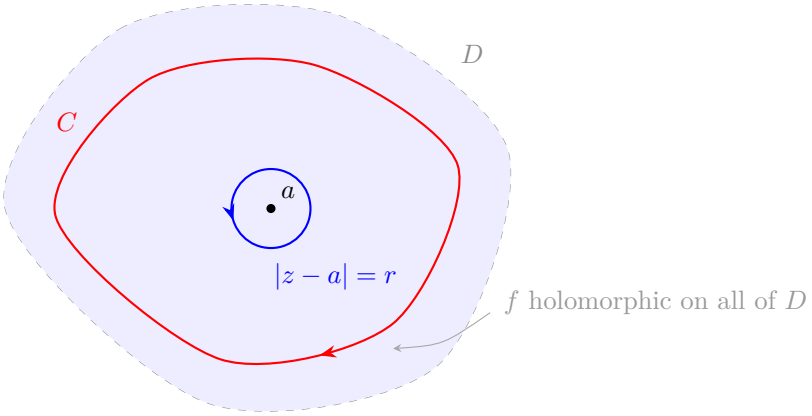


Figure 6: The setting for Cauchy's integral formula. The function f is holomorphic throughout D; the only thing wrong with the integrand $f(z)/(z-a)$ inside C is the single point a, and deformation lets us shrink C down onto a small circle around it.

Consider

$$\oint_C \frac{f(z)}{z-a}\,\mathrm{d}z,$$

whose integrand is holomorphic everywhere inside C except at the single point a. The trick is to separate off the singular piece by adding and subtracting $f(a)$,

$$\frac{f(z)}{z-a} = \underbrace{\frac{f(z)-f(a)}{z-a}}_{g(z)} + f(a)\,\frac{1}{z-a}.$$

Now shrink the contour down onto the point a, and watch two things happen. First, the integral of g does not change as we shrink, since no singularity is crossed on the way in and Eq. (17) applies. Second, the integral is at most the size of the integrand times the length of the path, and the length $2\pi r$ goes to zero while the integrand stays finite: unlike $f(z)/(z-a)$, the function g does not blow up at a, because its numerator $f(z)-f(a)$ vanishes there at the same rate as its denominator, leaving the finite limit $f'(a)$. So

$$\left|\oint_C g(z)\,\mathrm{d}z\right| \;\le\; (\max|g|)\cdot 2\pi r \;\xrightarrow[r\to 0]{}\; 0.$$

A number that cannot change and can be made as small as we like is zero, so the first piece contributes nothing.

What remains carries the constant $f(a)$ out in front,

$$\oint_C \frac{f(z)}{z-a}\,\mathrm{d}z = f(a)\oint_C \frac{\mathrm{d}z}{z-a},$$

and shrinking C onto a tiny circle centred at a, we can parametrise by $z = a + re^{\mathrm{i}\theta}$ to get $\mathrm{d}z = \mathrm{i}re^{\mathrm{i}\theta}\mathrm{d}\theta$ and $z-a = re^{\mathrm{i}\theta}$, so the remaining integral is $\int_0^{2\pi}\mathrm{i}\,\mathrm{d}\theta = 2\pi\mathrm{i}$. Rearranging gives *Cauchy's integral formula*,

$$\boxed{f(a) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z)}{z-a}\,\mathrm{d}z.} \tag{18}$$

On the left is a single number, the value of f at one interior point. On the right is f along the entire boundary. The kernel $1/(z-a)$ is the device that performs the extraction, weighting the

boundary so heavily near a that everything else cancels and only $f(a)$ survives: a holomorphic function cannot hide its interior from its boundary.

Differentiating under the integral sign with respect to a, which only ever touches the kernel and never f, repeats the statement for every derivative,

$$\boxed{f^{(n)}(a) = \frac{n!}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z.} \tag{19}$$

The right-hand side is manifestly finite for every n.

8 Laurent series and residues

As you may have noticed, the kernel $1/(z-a)$ has now appeared twice: once in Eq. (11) as the only power of z with a nonvanishing loop integral, and again in Eq. (18) as the device that extracts a value from the boundary. It is therefore natural to organise functions around it.

If f is holomorphic on a disc about a then it has a Taylor series there, every term has a single-valued antiderivative, and any closed contour inside the disc gives zero.

Suppose instead that f has an *isolated singularity* at a, meaning that f is holomorphic on a punctured disc $0 < |z-a| < R$ but not at the centre. A Taylor series is unavailable, since no power series can converge at a point where the function does not exist, but holomorphy on the punctured disc is still enough to guarantee a convergent *Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n = \cdots + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots, \tag{20}$$

valid on that punctured disc. The Laurent series is a Taylor series allowed to run over negative powers as well. Those negative powers are collectively the *principal part*, and counting them classifies the singularity:

- **Removable.** No negative powers at all. The Laurent series is an ordinary Taylor series and the point was never really singular in the first place. The standard example is $f(z) = \sin z/z$, where the denominator looks like trouble at the origin, but expanding $\sin z = z - z^3/3! + z^5/5! - \cdots$ and dividing gives $1 - z^2/3! + z^4/5! - \cdots$.
- **Pole of order m.** Finitely many negative powers, the worst being $(z-a)^{-m}$, so the principal part has exactly m terms. A pole of order one is called *simple*, and simple poles are the case that propagators produce.
- **Essential.** Infinitely many negative powers, no finite m suffices. The classic examples are $e^{1/z}$ at the origin, whose expansion $1 + 1/z + 1/(2!z^2) + 1/(3!z^3) + \cdots$ never terminates, and $\sin(1/z)$. Fortunately, we will not meet these again in this QFT course.

8.1 The residue theorem

Now consider what a closed contour does to Eq. (20). Take a contour C encircling a once, positively, and lying inside the punctured disc, and integrate the series term by term. Every term is a power of $(z-a)$, and we already know what a loop does to a power of $(z-a)$:

$$\oint_C (z-a)^n\,\mathrm{d}z = 2\pi\mathrm{i}\,\delta_{n,-1}, \qquad n \in \mathbb{Z}, \tag{21}$$

with $\delta_{n,-1}$ the Kronecker delta. Feeding Eq. (20) into it,

$$\oint_C f(z)\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n \oint_C (z-a)^n\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n\, 2\pi\mathrm{i}\,\delta_{n,-1} = 2\pi\mathrm{i}\,a_{-1}, \tag{22}$$

which is the same manipulation as projecting out a single Fourier coefficient. The surviving coefficient a_{-1} is important enough to have its own name, the *residue* of f at a, written $\mathrm{Res}(f;a)$.

For a contour enclosing several singularities, deform it into a chain of small circles, one around each enclosed z_j. Each little circle returns $2\pi\mathrm{i}$ times its own residue by Eq. (22), and summing over every isolated singularity $z_1,\dots,z_k$ inside C gives the *residue theorem*,

$$\boxed{\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{j=1}^{k}\mathrm{Res}(f;z_j).} \tag{23}$$

8.2 Computing residues

Since the residue is one particular Laurent coefficient, all we need is a way to reach in and pull out a_{-1} without constructing the whole series. For a pole of order m at z_0 write

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k + \frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \cdots + \frac{a_{-m}}{(z-z_0)^m}.$$

Multiplying by $(z-z_0)^m$ clears every denominator and leaves an ordinary power series,

$$(z-z_0)^m f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^{k+m} + a_{-1}(z-z_0)^{m-1} + a_{-2}(z-z_0)^{m-2} + \cdots + a_{-m},$$

in which the coefficient we want sits in front of $(z-z_0)^{m-1}$. Differentiating $m-1$ times annihilates every lower power and leaves a_{-1} multiplied by $(m-1)!$, while every higher power keeps a positive power of $(z-z_0)$ and dies in the limit $z\to z_0$. Hence

$$\boxed{\mathrm{Res}(f;z_0) = \lim_{z\to z_0}\frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\big[(z-z_0)^m f(z)\big].} \tag{24}$$

Almost every case we meet in practice is the simple pole $m=1$, where there is nothing to differentiate and the formula collapses to

$$\mathrm{Res}(f;z_0) = \lim_{z\to z_0}(z-z_0)f(z). \tag{25}$$

If $f = g/h$ with g and h holomorphic near a, and $h(a)=0$ with $h'(a)\neq 0$, so that a is a simple zero of the denominator, then expanding $h(z) = h'(a)(z-a) + O\big((z-a)^2\big)$ inside Eq. (25) gives

$$\boxed{\mathrm{Res}\Big(\frac{g}{h};a\Big) = \frac{g(a)}{h'(a)}.} \tag{26}$$

9 Closing contours on the real axis

There is one obvious obstacle left. The residue theorem Eq. (23) speaks about closed contours, and the integrals we actually want to do in field theory run along the whole real axis, which is

about as open as a contour can be. The strategy is to close them artificially and then argue that the extra piece was free,

$$\oint_C f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{\text{arc}} f(z)\,\mathrm{d}z, \tag{27}$$

where the arc is a big semicircle of radius R joining $+R$ back to $-R$ through one of the two half-planes. If we can show that the arc contributes nothing as $R \to \infty$, then the real integral we wanted equals the closed one, which by Eq. (23) is $\pm 2\pi\mathrm{i}$ times the enclosed residues, the sign being fixed by which way round the closed path ends up running. Everything therefore depends on controlling the arc, and there are two mechanisms for doing so.

Mechanism 1: polynomial decay. On the arc $z = Re^{\mathrm{i}\phi}$ the arc length grows like πR, so by the bound Eq. (7) the whole arc contribution is no bigger than $\pi R \max|f|$. If $|f|$ falls off at least as fast as $1/R^2$, the product behaves like $\pi R/R^2 \to 0$ and we are done.

Mechanism 2: exponential damping. Our motivating integral Eq. (1) carries the factor $e^{-\mathrm{i}k^0 t}$, which has modulus one everywhere on the real axis and therefore decays nowhere on it. Off the axis it is a different story. Write $k^0 = x + \mathrm{i}y$; the factor $e^{-\mathrm{i}xt}$ has modulus one and does nothing at all, so

$$\left|e^{-\mathrm{i}k^0 t}\right| = e^{yt}. \tag{28}$$

Damping means $e^{yt} < 1$, which is to say $yt < 0$: the imaginary part of k^0 must carry the opposite sign to t,

$$\boxed{t > 0 \implies \text{close below.} \qquad t < 0 \implies \text{close above.}} \tag{29}$$

The precise statement that the arc really does drop out, provided f itself tends to zero along it, is *Jordan's lemma*, and we will use it without proof.

Orientation then does the sign bookkeeping. Closing above traverses the loop counterclockwise and gives $+2\pi\mathrm{i}\sum \mathrm{Res}$, while closing below runs clockwise and gives $-2\pi\mathrm{i}\sum \mathrm{Res}$.

9.1 Polynomial decay in practice

Take $A > 0$ and evaluate

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^2 + A^2}.$$

The integrand extends into the plane as $1/(z^2 + A^2) = 1/((z - \mathrm{i}A)(z + \mathrm{i}A))$, with simple poles at $z = \pm\mathrm{i}A$, one in each half-plane. It falls off like $1/R^2$, so mechanism 1 applies and we may close whichever way we like, as in Figure 7.

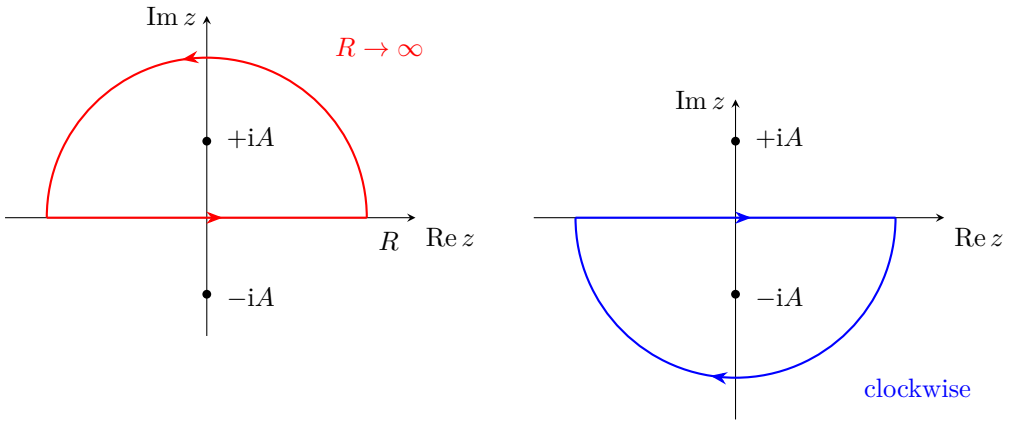
Closing above encloses only $z = +\mathrm{i}A$, and using Eq. (26) with $g = 1$ and $h = z^2 + A^2$, so that $h' = 2z$,

$$\mathrm{Res}(f; \mathrm{i}A) = \frac{1}{2\mathrm{i}A}, \qquad \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^2 + A^2} = 2\pi\mathrm{i} \cdot \frac{1}{2\mathrm{i}A} = \frac{\pi}{A}.$$

Closing below instead encloses $z = -\mathrm{i}A$ with residue $-1/(2\mathrm{i}A)$, and the clockwise orientation supplies a compensating minus sign,

$$(-2\pi\mathrm{i}) \cdot \left(-\frac{1}{2\mathrm{i}A}\right) = \frac{\pi}{A},$$

which agrees.



(a) close above: encloses $+\mathrm{i}A$ (b) close below: encloses $-\mathrm{i}A$

Figure 7: Two ways of closing the same integral when the integrand decays like $1/R^2$. The large arc contributes nothing in either case, and the two routes catch different poles with opposite orientations.

9.2 Exponential damping in practice

Now take

$$I = \int_{-\infty}^{+\infty} \frac{\cos x}{x^2 + A^2}\,\mathrm{d}x, \qquad A > 0.$$

One cannot close a contour with $\cos z$ sitting in the integrand, since $\cos z$ grows exponentially in *both* half-planes and neither closure is damped. Split it into exponentials first, and only then choose contours,

$$\cos x = \tfrac{1}{2}(e^{\mathrm{i}x} + e^{-\mathrm{i}x}) \qquad \Longrightarrow \qquad I = \tfrac{1}{2}\Big[\underbrace{\int \frac{e^{\mathrm{i}x}\,\mathrm{d}x}{x^2 + A^2}}_{=:I_1} + \underbrace{\int \frac{e^{-\mathrm{i}x}\,\mathrm{d}x}{x^2 + A^2}}_{=:I_2}\Big],$$

and treat each piece with mechanism 2. Each exponential selects its own half-plane, as in Figure 8.

For I_1 the exponent is $e^{+\mathrm{i}z}$, damped in the upper half-plane, so we close above and pick up $z = \mathrm{i}A$,

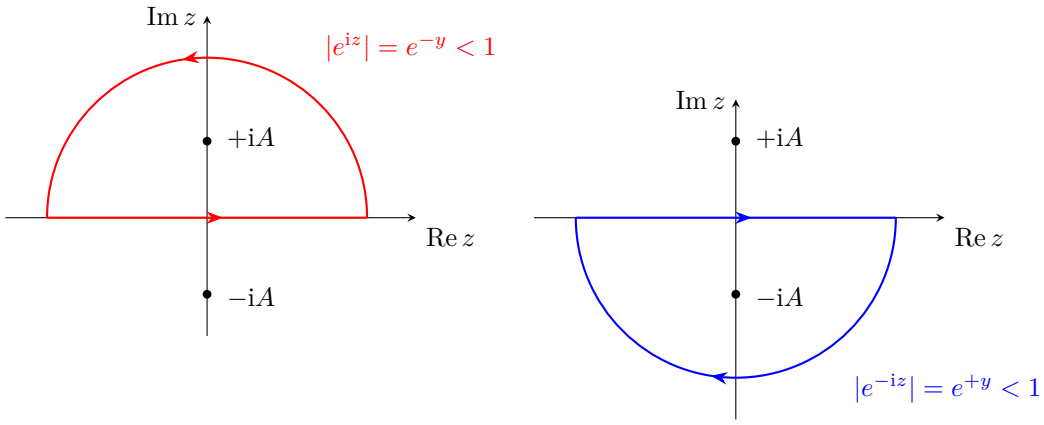
$$\operatorname{Res}\Big(\frac{e^{\mathrm{i}z}}{z^2 + A^2}; \mathrm{i}A\Big) = \frac{e^{\mathrm{i}(\mathrm{i}A)}}{2\mathrm{i}A} = \frac{e^{-A}}{2\mathrm{i}A}, \qquad I_1 = 2\pi\mathrm{i} \cdot \frac{e^{-A}}{2\mathrm{i}A} = \frac{\pi}{A}e^{-A}.$$

For I_2 the exponent is $e^{-\mathrm{i}z}$, so we close below and pick up $z = -\mathrm{i}A$, with the clockwise minus sign,

$$\operatorname{Res}\Big(\frac{e^{-\mathrm{i}z}}{z^2 + A^2}; -\mathrm{i}A\Big) = \frac{e^{-A}}{-2\mathrm{i}A}, \qquad I_2 = -2\pi\mathrm{i} \cdot \frac{e^{-A}}{-2\mathrm{i}A} = \frac{\pi}{A}e^{-A}.$$

Combining,

$$I = \tfrac{1}{2}(I_1 + I_2) = \boxed{\frac{\pi}{A}e^{-A}}.$$



(a) I_1, with $e^{+\mathrm{i}z}$: close above

(b) I_2, with $e^{-\mathrm{i}z}$: close below

Figure 8: The two halves of $\cos x$ are sent to opposite half-planes by the damping condition Eq. (28). Trying to keep them together, as $\cos z$, would leave us with an integrand that blows up whichever way we close.

9.3 The integral we started with

Everything is now in place for Eq. (1),

$$I(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \, \frac{e^{-\mathrm{i}k^0 t}}{(k^0)^2 - \omega_k^2 + \mathrm{i}\epsilon}.$$

Locate the poles first. The denominator vanishes when $(k^0)^2 = \omega_k^2 - \mathrm{i}\epsilon$, so

$$k^0 = \pm\sqrt{\omega_k^2 - \mathrm{i}\epsilon} \simeq \pm(\omega_k - \mathrm{i}\varepsilon), \qquad \varepsilon \equiv \frac{\epsilon}{2\omega_k} > 0,$$

and it is convenient to write down the factorized form

$$(k^0)^2 - \omega_k^2 + \mathrm{i}\epsilon \simeq (k^0 - \omega_k + \mathrm{i}\varepsilon)(k^0 + \omega_k - \mathrm{i}\varepsilon),$$

the two sides agreeing up to a term of order ϵ^2 which disappears in the limit anyway. One pole sits just *below* the real axis at $k^0 = +\omega_k - \mathrm{i}\varepsilon$, and the other just *above* it at $k^0 = -\omega_k + \mathrm{i}\varepsilon$. So the $\mathrm{i}\epsilon$ has taken the two bad points that were sitting on our path and pushed them off it in opposite directions, which is what makes the integral well defined in the first place. The two resulting contours are drawn in Figure 9.

Take $t > 0$ first. By Eq. (29) we close below, the loop runs clockwise, and the only pole inside is $k^0 = +\omega_k - \mathrm{i}\varepsilon$. Writing $a \equiv \omega_k - \mathrm{i}\varepsilon$, the integrand reads

$$\frac{e^{-\mathrm{i}k^0 t}}{(k^0 - a)(k^0 + a)}.$$

The residue at $k^0 = a$ now follows from Eq. (25):

$$\lim_{k^0 \to a} (k^0 - a) \, \frac{e^{-\mathrm{i}k^0 t}}{(k^0 - a)(k^0 + a)} = \lim_{k^0 \to a} \frac{e^{-\mathrm{i}k^0 t}}{k^0 + a} = \frac{e^{-\mathrm{i}at}}{2a}.$$

Together with the prefactor $1/2\pi$ and the factor $-2\pi\mathrm{i}$ supplied by the clockwise orientation,

$$I(t > 0) = \frac{1}{2\pi} \, (-2\pi\mathrm{i}) \, \frac{e^{-\mathrm{i}at}}{2a} \quad \xrightarrow[\varepsilon \to 0]{} \quad -\frac{\mathrm{i}}{2\omega_k} \, e^{-\mathrm{i}\omega_k t}.$$

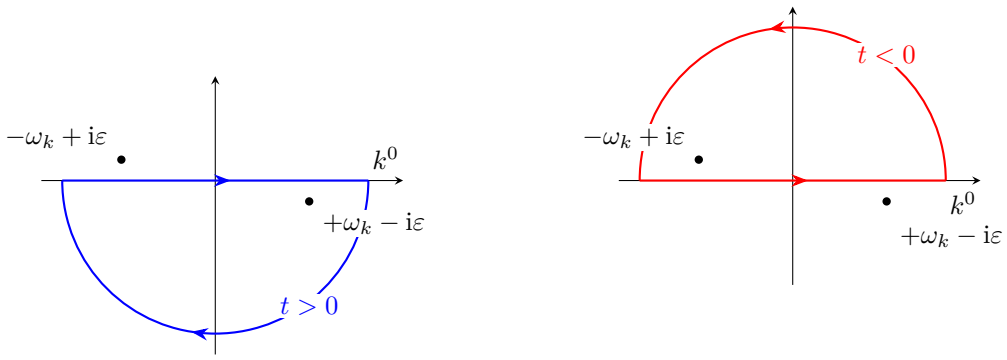


Figure 9: The two poles of $I(t)$ straddle the real axis. For $t > 0$ the damping forces us below, catching only $+\omega_k - \mathrm{i}\varepsilon$. For $t < 0$ it forces us above, catching only $-\omega_k + \mathrm{i}\varepsilon$.

For $t < 0$ we close above, the loop runs counterclockwise, and the enclosed pole is $k^0 = -a$, whose residue is $e^{+\mathrm{i}at}/(-2a)$. Then

$$I(t<0) = \frac{1}{2\pi}\,(2\pi\mathrm{i})\,\frac{e^{\mathrm{i}at}}{-2a} \;\xrightarrow[\varepsilon\to 0]{}\; -\frac{\mathrm{i}}{2\omega_k}\,e^{+\mathrm{i}\omega_k t}.$$

The two cases differ only by the sign in the exponent, so they combine into one expression,

$$\boxed{I(t) = -\frac{\mathrm{i}}{2\omega_k}\,e^{-\mathrm{i}\omega_k|t|}} \tag{30}$$

10 Propagators

10.1 From $I(t)$ to the Feynman propagator

The integral Eq. (1) is in reality closely related the scalar propagator. For a four-momentum $k^\mu = (k^0, \vec{k})$ we write

$$\omega_k \equiv +\sqrt{\vec{k}^{\,2} + m^2}, \qquad \text{so that} \qquad k^2 + m^2 = \omega_k^2 - (k^0)^2.$$

A four-momentum obeying $k^2 + m^2 = 0$, equivalently $k^0 = \pm\omega_k$, is said to be *on shell*, that condition being the mass-shell hyperboloid on which a real propagating particle must sit. Inside a propagator, however, k^0 is an integration variable and ranges over the whole real line, on shell and off.

The Feynman propagator for a scalar of mass m is

$$G_F(x,y) = \langle 0|\mathrm{T}\{\phi(x)\phi(y)\}|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{-\mathrm{i}\,e^{\mathrm{i}k_\mu(x-y)^\mu}}{k^2 + m^2 - \mathrm{i}\epsilon}. \tag{31}$$

Two substitutions turn its k^0 integral into $I(t)$. First the exponent: with $k_\mu(x-y)^\mu = -k^0 t + \vec{k}\cdot\vec{r}$, where $t \equiv x^0 - y^0$ and $\vec{r} \equiv \vec{x} - \vec{y}$,

$$e^{\mathrm{i}k_\mu(x-y)^\mu} = e^{-\mathrm{i}k^0 t}\,e^{\mathrm{i}\vec{k}\cdot\vec{r}},$$

which supplies precisely the damping factor Eq. (28). Second the denominator: since $k^2 + m^2 = \omega_k^2 - (k^0)^2$,

$$k^2 + m^2 - \mathrm{i}\epsilon = -\Big[(k^0)^2 - \omega_k^2 + \mathrm{i}\epsilon\Big],$$

so the Feynman denominator is minus the denominator of $I(t)$, and the iϵ prescriptions match. Collecting the factors of $-\mathrm{i}$ and -1 and using Eq. (30),

$$\int \frac{\mathrm{d}k^0}{2\pi}\,\frac{-\mathrm{i}\,e^{-\mathrm{i}k^0 t}}{k^2+m^2-\mathrm{i}\epsilon} = \mathrm{i}\,I(t) = \boxed{\frac{1}{2\omega_k}\,e^{-\mathrm{i}\omega_k|t|}} \tag{32}$$

It is worth looking back at what the iϵ actually did. In Eq. (1) it looked like a small number whose only job was to keep the integrand finite on the axis, and it did do that. But it did something else besides: it decided which side of the axis each pole sits on. The pole at $+\omega_k$ was pushed below and the one at $-\omega_k$ above, and since the sign of t dictates which way we are allowed to close, each pole is attached to one sign of t.

Writing the exponent of Eq. (32) as its two cases,

$$e^{-\mathrm{i}\omega_k|t|} = \theta(t)\,e^{-\mathrm{i}\omega_k t} + \theta(-t)\,e^{+\mathrm{i}\omega_k t}, \tag{33}$$

each term comes from one pole: the first from the pole below the axis, which only $t>0$ can reach, and the second from the pole above, which only $t<0$ can reach.

Now compare with the formal definition of the Feynman propagator as a time-ordered vacuum expectation value, $G_F(x,y) = \langle 0|\mathrm{T}\{\phi(x)\phi(y)\}|0\rangle$, in which the time-ordering symbol is itself defined by a pair of θ functions,

$$\mathrm{T}\{\phi(x)\phi(y)\} = \theta(x^0-y^0)\,\phi(x)\phi(y) + \theta(y^0-x^0)\,\phi(y)\phi(x). \tag{34}$$

Since $t = x^0 - y^0$, the two θ functions produced by the contour in Eq. (33) are the two appearing in Eq. (34), and each term, once the remaining d^3k integration is done, assembles into one of the two orderings of the fields. Writing the denominator as $k^2+m^2-\mathrm{i}\epsilon$ and integrating over all real k^0 therefore does the same job as the operator T does in the definition: the iϵ prescription is what time ordering looks like in momentum space.

Had the iϵ placed both poles on the same side of the axis, the closure for one sign of t would have enclosed nothing at all, one of the two terms in Eq. (33) would be missing, and the result would vanish for that sign of t instead of describing the opposite ordering; those are the retarded and advanced propagators as in the Problem Set 2. Which orderings a propagator contains is fixed entirely by where the iϵ puts the poles.

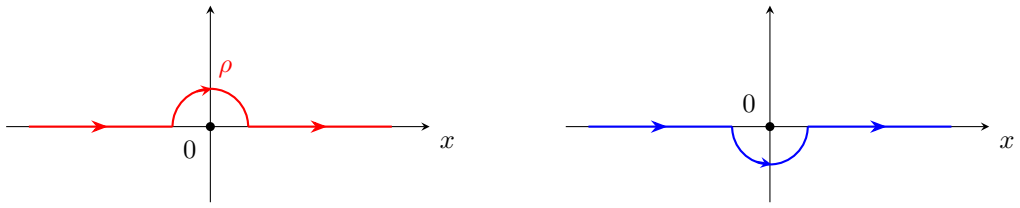
10.2 Poles sitting on the contour: the principal value

One last case. Sometimes a pole sits exactly on the real axis with no iϵ to tell us which way to go around it. Strictly speaking the integral does not exist, so the resolution is to record both possible answers and see how they differ. Deform the contour into a small semicircle of radius ρ around the pole, either over it or under it, and let $\rho \to 0$, as in Figure 10.

The straight parts assemble into the *principal value* P, which is by definition the symmetric limit that omits a shrinking interval around the pole, and the semicircle contributes half of a full residue, since it is half a circle, with a sign set by which way it detours. Writing the two choices as the two iϵ prescriptions that produce them gives the Sokhotski–Plemelj relation (link),

$$\boxed{\frac{1}{x\pm\mathrm{i}\epsilon} = \mathrm{P}\frac{1}{x} \mp \mathrm{i}\pi\,\delta(x).} \tag{35}$$

So an iϵ turns into a delta function, which is the mechanism behind decay rates and optical theorems later in the course. It also says precisely how the two prescriptions we met above differ from one another: they share the same principal-value part, and differ only by delta functions supported on the mass shell.



(a) over the pole: $\mathrm{P} - \mathrm{i}\pi\,\mathrm{Res}$ (b) under the pole: $\mathrm{P} + \mathrm{i}\pi\,\mathrm{Res}$

Figure 10: A pole sitting on the integration path. The straight parts assemble into the principal value, and the little semicircle supplies half of a residue, with a sign depending on which way we chose to walk around the pole. The difference between the two choices is a whole residue, that is, a delta function.