Section 3: Gaussian Integrals

Wednesday, September 23rd 2026

Srednicki, Ch. 7–8 + Wiki on Gaussian integrals, notes from PHYS262 2025

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1 Why and where Gaussian integrals?

In lecture the vacuum-to-vacuum amplitude of a scalar field ϕ in the presence of a source $J(x)$ was written as a path integral, and every time-ordered correlation function was obtained from it by differentiating with respect to the source:

$$Z[J] = \langle 0|0\rangle_J = \int \mathcal{D}\phi \; e^{i\int d^4x\,(\mathcal{L}+J\phi)}, \quad \langle 0|\,\mathrm{T}\{\phi(x_1)\cdots\phi(x_n)\}\,|0\rangle = \frac{1}{i^n Z[0]}\,\frac{\delta^n Z[J]}{\delta J_1\cdots\delta J_n}\bigg|_{J=0}, \tag{1}$$

where $J_i \equiv J(x_i)$. This is a compact and complete statement of the theory, and it hides one enormous practical question: how does one actually *do* the integral $\int \mathcal{D}\phi$? It runs over all functions ϕ on spacetime, so it has, in effect, one integration variable for every point of spacetime. Very few such integrals can be done. As it happens, the ones this course needs all belong to a single family of such integrals.

Example: the free theory is a Gaussian. The free scalar has $\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$. Integrating the kinetic term by parts, with $\Box \equiv \partial_\mu\partial^\mu = -\partial_t^2 + \nabla^2$ and dropping the boundary term,

$$-\frac{1}{2}\int d^4x\;\partial_\mu\phi\,\partial^\mu\phi = -\frac{1}{2}\int d^4x\Big[\partial_\mu(\phi\,\partial^\mu\phi) - \phi\,\Box\phi\Big] = \frac{1}{2}\int d^4x\;\phi\,\Box\phi,$$

so that the exponent of $Z[J]$ for the free theory is

$$S[\phi] + \int d^4x\,J\phi = \underbrace{-\frac{1}{2}\int d^4x\;\phi\,(-\Box + m^2)\,\phi}_{\text{quadratic form in }\phi} + \underbrace{\int d^4x\;J\,\phi}_{\text{linear in }\phi}. \tag{2}$$

Compare with the simplest possible integral of the same shape, in one variable x,

$$\int_{-\infty}^{+\infty} dx\; e^{i\left(-\frac{1}{2}ax^2+bx\right)}, \qquad \phi \leftrightarrow x, \qquad (-\Box + m^2) \leftrightarrow a, \qquad J \leftrightarrow b. \tag{3}$$

An exponential of a quadratic form plus a linear term is called a *Gaussian*, and $Z[J]$ is a Gaussian integral over infinitely many variables.

2 One variable

2.1 The basic integral

Consider

$$I(a) = \int_{-\infty}^{+\infty} dx\; e^{-\frac{1}{2}ax^2}, \qquad a > 0. \tag{4}$$

The condition $a > 0$ is needed for the integrand to decay at large $|x|$; for $a < 0$ the integral is infinite. The factor $\frac{1}{2}$ is a convention that makes a the coefficient of the quadratic form in the same sense as $m\omega^2$ in $\frac{1}{2}m\omega^2 q^2$. The integrand is the familiar bell curve (Figure 1), and $I(a)$ is the area underneath it.

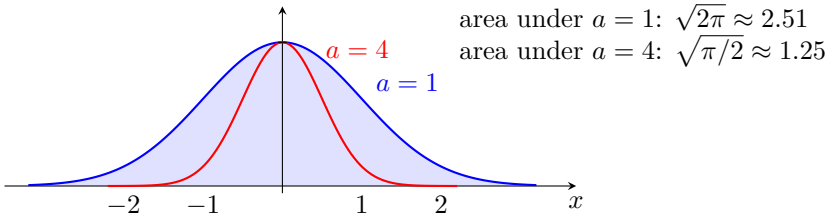


Figure 1: The integrand $e^{-\frac{1}{2}ax^2}$ for two values of a. A larger a makes a narrower peak and a smaller area; the area scales as $a^{-1/2}$.

There is no elementary antiderivative of $e^{-\frac{1}{2}ax^2}$. The trick is to compute the *square* of the integral instead, which is a double integral over the plane, and to do that double integral in polar coordinates:

$$I(a)^2 = \int_{-\infty}^{+\infty} dx\; e^{-\frac{1}{2}ax^2} \int_{-\infty}^{+\infty} dy\; e^{-\frac{1}{2}ay^2} = \iint dx\, dy\; e^{-\frac{1}{2}a(x^2+y^2)} = \left| \begin{array}{l} x = r\cos\theta, \quad y = r\sin\theta \\ dx\, dy = r\, dr\, d\theta \\ x^2 + y^2 = r^2 \\ r \in [0,\infty),\ \ \theta \in [0, 2\pi) \end{array} \right|$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} r\, dr\; e^{-\frac{1}{2}ar^2} = \left| \begin{array}{l} u = \frac{1}{2}ar^2 \\ du = a\, r\, dr \end{array} \right| = 2\pi \int_0^{\infty} \frac{du}{a}\, e^{-u} = \frac{2\pi}{a}\Big[-e^{-u}\Big]_0^{\infty} = \frac{2\pi}{a}.$$

Taking the positive square root,

$$\boxed{\int_{-\infty}^{+\infty} dx\; e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}} \tag{5}$$

2.2 Completing the square

Now add a linear term to the exponent. The idea is to rewrite the exponent as a perfect square plus a constant, so that the linear term disappears into a shift of the variable:

$$-\frac{1}{2}ax^2 + bx = -\frac{1}{2}a\Big(x^2 - \frac{2b}{a}x\Big) = -\frac{1}{2}a\Big[\Big(x - \frac{b}{a}\Big)^2 - \frac{b^2}{a^2}\Big] = -\frac{1}{2}a\Big(x - \frac{b}{a}\Big)^2 + \frac{b^2}{2a}. \tag{6}$$

The constant $b^2/2a$ comes out of the integral, and the shifted square is Eq. (5) in a new variable:

$$Z(b) \equiv \int_{-\infty}^{+\infty} dx\; e^{-\frac{1}{2}ax^2 + bx} = e^{b^2/2a} \int_{-\infty}^{+\infty} dx\; e^{-\frac{1}{2}a\left(x - \frac{b}{a}\right)^2} = \left| \begin{array}{l} y = x - b/a \\ dy = dx \\ \text{limits } \pm\infty \text{ unchanged} \end{array} \right|$$

$$= e^{b^2/2a} \int_{-\infty}^{+\infty} dy\; e^{-\frac{1}{2}ay^2},$$

so that

$$\boxed{Z(b) = \int_{-\infty}^{+\infty} dx\; e^{-\frac{1}{2}ax^2+bx} = \sqrt{\frac{2\pi}{a}}\; e^{\,b^2/2a}} \tag{7}$$

Complex b. Nothing in the derivation required b to be real, but if $b = \beta + i\gamma$ the shift $y = x - b/a$ moves the integration path off the real axis to the horizontal line $\operatorname{Im} y = -\gamma/a$ (Figure 2). That the integral over the shifted line equals the integral over the real axis is a contour argument. Close the two lines into a rectangle with the vertical segments at $\operatorname{Re} y = \pm R$. The integrand $e^{-\frac{1}{2}ay^2}$ has no singularities anywhere, so by Cauchy's theorem the integral around the closed rectangle vanishes,

$$\int_{-R}^{R} dy\; e^{-\frac{1}{2}ay^2} + \int_{\text{right side}} dy\; e^{-\frac{1}{2}ay^2} - \int_{-R-i\gamma/a}^{R-i\gamma/a} dy\; e^{-\frac{1}{2}ay^2} - \int_{\text{left side}} dy\; e^{-\frac{1}{2}ay^2} = 0.$$

On the vertical sides $y = \pm R + is$ with s between 0 and $-\gamma/a$, and

$$\left|e^{-\frac{1}{2}a(\pm R+is)^2}\right| = \left|e^{-\frac{1}{2}a(R^2-s^2\pm 2iRs)}\right| = e^{-\frac{1}{2}a(R^2-s^2)} \xrightarrow[R\to\infty]{} 0$$

uniformly in s on the finite segment, so both side integrals vanish as $R \to \infty$ and the two horizontal line integrals agree. Eq. (7) therefore holds for all complex b.

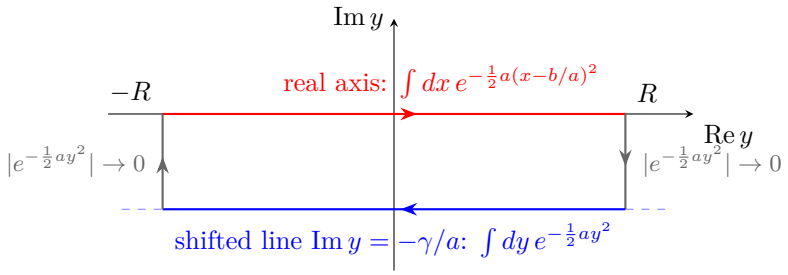


Figure 2: The contour behind the shift $y = x - b/a$ for complex $b = \beta + i\gamma$ (drawn for $\gamma/a > 0$). The original integral runs along the real axis, the shifted one along the line $\operatorname{Im} y = -\gamma/a$. Closing the rectangle with two vertical segments at $\operatorname{Re} y = \pm R$ encloses no singularity, and the vertical segments contribute nothing as $R \to \infty$, so the two horizontal integrals are equal.

In particular, with $b = ik$,

$$\int_{-\infty}^{+\infty} dx\; e^{-\frac{1}{2}ax^2+ikx} = \sqrt{\frac{2\pi}{a}}\; e^{-k^2/2a}\,: \tag{8}$$

the Fourier transform of a Gaussian of width $1/\sqrt{a}$ is a Gaussian of width $\sqrt{a}$.

2.3 Oscillatory Gaussians

In quantum theory the exponent is iS, not $-S$, so the integral we actually meet has an imaginary coefficient in front of the square,

$$\int_{-\infty}^{+\infty} dx\; e^{\frac{i}{2}ax^2}, \qquad a \in \mathbb{R},\ a \neq 0. \tag{9}$$

The integrand has modulus one everywhere, so the integral does not converge absolutely. What saves it is that the phase $\frac{1}{2}ax^2$ turns faster and faster as $|x|$ grows (Figure 3), so that neighbouring half-waves cancel more and more precisely. The integral is conditionally convergent, and there are two standard ways to compute it; they agree, and both will be used later.

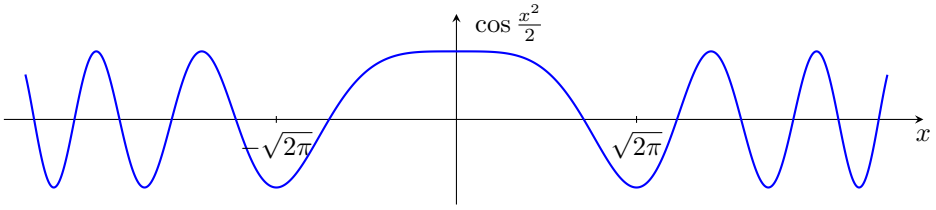


Figure 3: The real part of the integrand $e^{ix^2/2}$. Near the origin the phase is stationary and the contributions add up; far away the oscillations become ever faster and cancel in pairs. The whole integral is dominated by the region $|x| \lesssim \sqrt{2\pi/a}$.

Method 1: a small damping and analytic continuation. Multiply the integrand by $e^{-\frac{1}{2}\epsilon x^2}$ with $\epsilon > 0$, which makes the integral absolutely convergent, and remove the damping at the end. The damped integral is of the form Eq. (5) with a complex coefficient:

$$\int_{-\infty}^{+\infty} dx\, e^{\frac{i}{2}ax^2 - \frac{1}{2}\epsilon x^2} = \int_{-\infty}^{+\infty} dx\, e^{-\frac{1}{2}cx^2}, \qquad c \equiv \epsilon - ia, \qquad \operatorname{Re} c = \epsilon > 0.$$

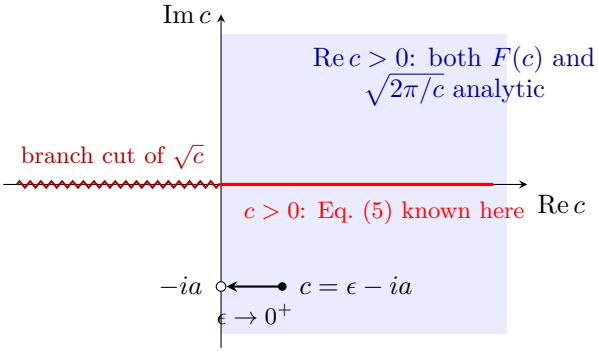


Figure 4: The complex c-plane in Method 1 (drawn for $a > 0$). The formula $\int dx\, e^{-\frac{1}{2}cx^2} = \sqrt{2\pi/c}$ is derived on the positive real axis (thick red line) and extends by analyticity to the whole shaded half-plane $\operatorname{Re} c > 0$, where the integral converges and the principal square root has no cut. The oscillatory Gaussian is the limit in which $c = \epsilon - ia$ approaches the point $-ia$ on the boundary from inside the region.

Eq. (5) was derived for real positive c. To extend it, consider $F(c) = \int dx\, e^{-\frac{1}{2}cx^2}$ as a function of the complex variable c on the half-plane $\operatorname{Re} c > 0$ (Figure 4). There $|e^{-\frac{1}{2}cx^2}| = e^{-\frac{1}{2}(\operatorname{Re} c)x^2}$, so the integral converges absolutely, and differentiating under the integral sign shows that F is analytic in c. The function $\sqrt{2\pi/c}$, with the square root on its principal branch ($-\pi < \arg c < \pi$, cut along the negative real axis), is also analytic on $\operatorname{Re} c > 0$, since the cut lies outside. Two analytic functions that coincide on the positive real axis coincide on the whole connected region where both are analytic. Hence

$$\int_{-\infty}^{+\infty} dx\, e^{-\frac{1}{2}cx^2} = \sqrt{\frac{2\pi}{c}} \qquad \text{for all } \operatorname{Re} c > 0, \quad \text{principal square root.} \tag{10}$$

Now set $c = \epsilon - ia$ and let $\epsilon \to 0^+$. The point $-ia$ lies on the boundary of the half-plane, and $\sqrt{2\pi/c}$ is continuous up to it, so

$$\int_{-\infty}^{+\infty} dx\, e^{\frac{i}{2}ax^2} = \lim_{\epsilon \to 0^+} \sqrt{\frac{2\pi}{\epsilon - ia}} = \sqrt{\frac{2\pi}{-ia}}.$$

To evaluate the square root write $-ia$ in polar form: for $a>0$, $-ia = a\,e^{-i\pi/2}$, and for $a<0$, $-ia = |a|\,e^{+i\pi/2}$; in both cases $-ia = |a|\,e^{-i\frac{\pi}{2}\mathrm{sgn}(a)}$, so

$$\sqrt{-ia} = \sqrt{|a|}\;e^{-i\frac{\pi}{4}\mathrm{sgn}(a)}, \qquad \sqrt{\frac{2\pi}{-ia}} = \sqrt{\frac{2\pi}{|a|}}\;e^{+i\frac{\pi}{4}\mathrm{sgn}(a)}.$$

Therefore

$$\boxed{\int_{-\infty}^{+\infty} dx\; e^{\frac{i}{2}ax^2} = \sqrt{\frac{2\pi}{-ia}} = \sqrt{\frac{2\pi}{|a|}}\;e^{\,i\frac{\pi}{4}\mathrm{sgn}(a)}} \tag{11}$$

Method 2: rotating the contour. Take $a>0$. The substitution $x = e^{i\pi/4}u$ would turn the oscillating exponent into a decaying one, because $x^2 = e^{i\pi/2}u^2 = iu^2$ and so $\frac{i}{2}ax^2 = \frac{i}{2}a\,iu^2 = -\frac{1}{2}au^2$. The question is whether we are allowed to integrate along the line $x = e^{i\pi/4}u$, $u\in\mathbb{R}$, instead of along the real axis. Consider the closed contour in Figure 5: from 0 to R along the real axis, then along the arc C_R, $x = Re^{i\theta}$ with θ from 0 to $\pi/4$, then back to the origin along the ray $x = e^{i\pi/4}u$ with u from R to 0. The integrand $e^{\frac{i}{2}ax^2}$ is analytic everywhere, so Cauchy's theorem gives

$$\int_0^R dx\; e^{\frac{i}{2}ax^2} + \int_{C_R} dx\; e^{\frac{i}{2}ax^2} - \int_0^R e^{i\pi/4}du\; e^{\frac{i}{2}a(e^{i\pi/4}u)^2} = 0. \tag{12}$$

The arc contribution vanishes as $R\to\infty$. On C_R, $x^2 = R^2e^{2i\theta}$ and

$$\left|e^{\frac{i}{2}aR^2e^{2i\theta}}\right| = \left|e^{\frac{i}{2}aR^2(\cos 2\theta + i\sin 2\theta)}\right| = e^{-\frac{1}{2}aR^2\sin 2\theta}, \qquad |dx| = R\,d\theta,$$

so that, using the inequality $\sin 2\theta \geq \frac{4\theta}{\pi}$ for $0\leq\theta\leq\frac{\pi}{4}$,

$$\left|\int_{C_R} dx\; e^{\frac{i}{2}ax^2}\right| \leq \int_0^{\pi/4} R\,d\theta\; e^{-\frac{1}{2}aR^2\sin 2\theta} \leq \int_0^{\pi/4} R\,d\theta\; e^{-\frac{2aR^2}{\pi}\theta} = \frac{\pi}{2aR}\Big(1 - e^{-\frac{1}{2}aR^2}\Big) \xrightarrow[R\to\infty]{} 0,$$

With the arc gone, Eq. (12) says that the integral along the positive real axis equals the integral along the ray, and the latter is a real Gaussian:

$$\int_0^\infty dx\; e^{\frac{i}{2}ax^2} = \int_0^\infty e^{i\pi/4}du\; e^{\frac{i}{2}a\,iu^2} = e^{i\pi/4}\int_0^\infty du\; e^{-\frac{1}{2}au^2} = e^{i\pi/4}\cdot\frac{1}{2}\sqrt{\frac{2\pi}{a}},$$

where the last step uses that the integrand is even, so the half-line carries half of Eq. (5). The same argument in the lower-left sector handles the negative real axis, or one simply uses evenness again, and

$$\int_{-\infty}^{+\infty} dx\; e^{\frac{i}{2}ax^2} = 2\int_0^\infty dx\; e^{\frac{i}{2}ax^2} = e^{i\pi/4}\sqrt{\frac{2\pi}{a}},$$

in agreement with Eq. (11) for $a>0$. For $a<0$ one rotates the other way, $x = e^{-i\pi/4}u$, and finds $e^{-i\pi/4}\sqrt{2\pi/|a|}$. Rotating the integration variable by 45° in the complex plane to turn a phase into a damping is a one-variable version of the *Wick rotation* $t\to -i\tau$ of the time coordinate, which turns e^{iS} into e^{-S_E}.

With a linear term. Completing the square works as before. For real $a\neq 0$,

$$\frac{i}{2}ax^2 + ibx = \frac{i}{2}a\Big(x^2 + \frac{2b}{a}x\Big) = \frac{i}{2}a\Big(x + \frac{b}{a}\Big)^2 - \frac{ib^2}{2a},$$

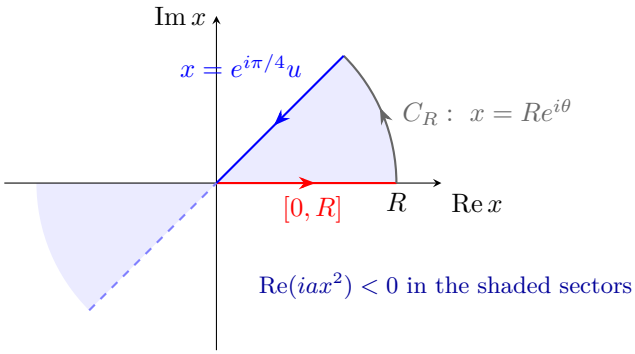


Figure 5: The contour used in Method 2 for $a > 0$. Inside the shaded sector the integrand $e^{\frac{i}{2}ax^2}$ decays, the arc C_R contributes nothing as $R \to \infty$, and the integral along the real axis may be traded for the integral along the line at 45°, where the Gaussian is real.

and shifting $x = y - b/a$ (allowed for the same contour reason as in the real case),

$$\boxed{\int_{-\infty}^{+\infty} dx\, e^{\frac{i}{2}ax^2+ibx} = e^{-\frac{ib^2}{2a}} \int_{-\infty}^{+\infty} dy\, e^{\frac{i}{2}ay^2} = \sqrt{\frac{2\pi}{-ia}}\, e^{-ib^2/2a}} \tag{13}$$

As a check, apply this to the momentum integral

$$\langle q_{j+1}|\, e^{-iH\delta t}\, |q_j\rangle = \int \frac{dp}{2\pi}\, \exp\Big[\, i\,p\,(q_{j+1} - q_j) - i\,\delta t \frac{p^2}{2m}\Big], \tag{14}$$

from the derivation of the path integral, ignoring the potential term which does not involve p. Its exponent is $-\frac{i\delta t}{2m}p^2 + i(q_{j+1} - q_j)p$, which is Eq. (13) in the variable p with

$$a = -\frac{\delta t}{m}, \qquad b = q_{j+1} - q_j \equiv \dot{q}\,\delta t.$$

Then $-ia = i\delta t/m$, so the prefactor is $\frac{1}{2\pi}\sqrt{2\pi m/(i\delta t)} = (m/2\pi i\,\delta t)^{1/2}$, and the exponent is

$$-\frac{ib^2}{2a} = -\frac{i(\dot{q}\,\delta t)^2}{2} \cdot \Big(-\frac{m}{\delta t}\Big) = \frac{i}{2}\, m\dot{q}^2\,\delta t,$$

which is $i\,\delta t$ times the kinetic term $\frac{1}{2}m\dot{q}^2$ of the Lagrangian. This is how $p\dot{q} - H$ turned into L.

2.4 Moments

Look at what happens when Eq. (7) is differentiated with respect to the source b. On the left-hand side the derivative acts on e^{bx} and brings down a factor of x each time:

$$\frac{dZ}{db} = \int dx\; x\; e^{-\frac{1}{2}ax^2+bx}, \qquad \frac{d^2Z}{db^2} = \int dx\; x^2\; e^{-\frac{1}{2}ax^2+bx}, \qquad \frac{d^nZ}{db^n} = \int dx\; x^n\; e^{-\frac{1}{2}ax^2+bx}. \tag{15}$$

Setting $b = 0$ afterwards, the right-hand sides become $\int dx\, x^n e^{-\frac{1}{2}ax^2}$, that is, Gaussian integrals with a power of x inserted.
It is convenient to divide by $Z(0)$, so that the weight $e^{-\frac{1}{2}ax^2}/Z(0)$ integrates to one and may be thought of as a probability density for x. The average of x^n with respect to that density is called the *n-th moment* of the Gaussian,

$$\langle x^n\rangle \equiv \frac{\int dx\; x^n\, e^{-\frac{1}{2}ax^2}}{\int dx\; e^{-\frac{1}{2}ax^2}} = \frac{1}{Z(0)}\, \frac{d^n}{db^n} Z(b)\Big|_{b=0}. \tag{16}$$

3 Several variables

3.1 From one variable to N: diagonalising the quadratic form

In field theory there is not one integration variable but one for every point of space, so the next step is to allow N variables $x = (x_1, \ldots, x_N)$ and ask for the most general Gaussian in them. "Quadratic in x" now means any expression of the form

$$\underbrace{\frac{1}{2}\sum_{i,j=1}^{N} x_i A_{ij} x_j}_{\text{quadratic form}} \equiv \frac{1}{2}\, x^{\mathsf{T}} A x, \qquad \text{e.g. for } N=2: \quad \frac{1}{2}(A_{11}x_1^2 + A_{12}x_1x_2 + A_{21}x_2x_1 + A_{22}x_2^2), \tag{17}$$

with A an $N \times N$ real matrix, and "linear in x" means $\sum_i b_i x_i = b^{\mathsf{T}} x$ for some vector b. Since $x_i x_j = x_j x_i$, only the combination $A_{ij} + A_{ji}$ enters the quadratic form: indeed $x^{\mathsf{T}} A x = (x^{\mathsf{T}} A x)^{\mathsf{T}} = x^{\mathsf{T}} A^{\mathsf{T}} x$, so $x^{\mathsf{T}} A x = x^{\mathsf{T}} \frac{A + A^{\mathsf{T}}}{2} x$, and we may and shall assume $A = A^{\mathsf{T}}$.

Start with the integral without sources, $\int d^N x\, e^{-\frac{1}{2} x^{\mathsf{T}} A x}$ with $d^N x = dx_1 \cdots dx_N$. If A happens to be diagonal, $A = \mathrm{diag}(a_1, \ldots, a_N)$, the exponent is a sum $-\frac{1}{2}\sum_n a_n x_n^2$, the exponential is a product, and the N-dimensional integral factorizes into N one-dimensional ones:

$$\int d^N x\; e^{-\frac{1}{2}\sum_n a_n x_n^2} = \prod_{n=1}^{N} \int dx_n\; e^{-\frac{1}{2} a_n x_n^2} = \prod_{n=1}^{N} \sqrt{\frac{2\pi}{a_n}}.$$

If A is not diagonal, the exponent contains cross terms like $A_{12}x_1x_2$ and does not factorize as simply. But a real symmetric matrix can always be diagonalized by an orthogonal change of variables, and an orthogonal change of variables costs nothing in a Gaussian integral. The spectral theorem says there is an orthogonal matrix O ($O^{\mathsf{T}} O = O O^{\mathsf{T}} = \mathbf{1}$) such that

$$O A O^{\mathsf{T}} = D = \mathrm{diag}(a_1, \ldots, a_N), \qquad \text{equivalently} \qquad A = O^{\mathsf{T}} D O, \tag{18}$$

where the a_n are the eigenvalues of A and the rows of O are the corresponding normalised eigenvectors. Then:

$$\int d^N x\; e^{-\frac{1}{2} x^{\mathsf{T}} A x} = \left| \begin{array}{l} y = Ox, \quad x = O^{\mathsf{T}} y \\ x^{\mathsf{T}} A x = y^{\mathsf{T}} O A O^{\mathsf{T}} y = y^{\mathsf{T}} D y = \sum_n a_n y_n^2 \\ d^N x = \left|\det \frac{\partial x}{\partial y}\right| d^N y = |\det O^{\mathsf{T}}|\, d^N y = d^N y, \text{ since } (\det O)^2 = \det(O^{\mathsf{T}} O) = 1 \end{array} \right|$$

$$= \int d^N y\; e^{-\frac{1}{2}\sum_n a_n y_n^2} = \prod_{n=1}^{N} \int dy_n\; e^{-\frac{1}{2} a_n y_n^2} = \prod_{n=1}^{N} \sqrt{\frac{2\pi}{a_n}} = \frac{(2\pi)^{N/2}}{\sqrt{a_1 a_2 \cdots a_N}}.$$

Finally, the product of the eigenvalues is the determinant of A, because the determinant of a product is the product of determinants and $\det O^{\mathsf{T}} \det O = 1$:

$$\det A = \det(O^{\mathsf{T}} D O) = \det O^{\mathsf{T}}\, \det D\, \det O = \det D = a_1 a_2 \cdots a_N.$$

Therefore

$$\boxed{\int d^N x\; e^{-\frac{1}{2} x^{\mathsf{T}} A x} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}} \tag{19}$$

This is Eq. (5) with the number a replaced by the matrix A, and "$a^{-1/2}$" replaced by "$(\det A)^{-1/2}$". Each of the N one-dimensional integrals required its own $a_n > 0$, so the condition for convergence is that all eigenvalues of A be positive, that is, A positive definite. If an eigenvalue a_n vanishes, the integrand does not depend on y_n at all, $\int dy_n\, 1 = \infty$, and the integral diverges; such a direction in x-space along which the exponent is flat is called a *zero mode*.

3.2 Sources, the inverse, and Wick's theorem

Now add the linear term and complete the square in N variables:

$$Z(b) \equiv \int d^N x\; e^{-\frac{1}{2}x^{\mathsf{T}}Ax + b^{\mathsf{T}}x}. \tag{20}$$

In one variable the shift was to the point $x_* = b/a$ where the exponent is stationary. Here the exponent is stationary where its gradient vanishes,

$$\frac{\partial}{\partial x_i}\Big(-\frac{1}{2}\sum_{kl} x_k A_{kl} x_l + \sum_k b_k x_k\Big) = -\frac{1}{2}\sum_l A_{il}x_l - \frac{1}{2}\sum_k x_k A_{ki} + b_i = -\sum_l A_{il}x_l + b_i,$$

where $A = A^{\mathsf{T}}$ was used to combine the two middle terms; setting this to zero gives $Ax_* = b$. Since $\det A \neq 0$ the matrix is invertible and $x_* = A^{-1}b$. Note that A^{-1} is symmetric too, because $(A^{-1})^{\mathsf{T}} = (A^{\mathsf{T}})^{-1} = A^{-1}$. Now substitute $x = x_* + y$ and expand the exponent:

$$\begin{aligned}
-\frac{1}{2}(x_*+y)^{\mathsf{T}}A(x_*+y) + b^{\mathsf{T}}(x_*+y) &= -\frac{1}{2}x_*^{\mathsf{T}}Ax_* - \frac{1}{2}x_*^{\mathsf{T}}Ay - \frac{1}{2}y^{\mathsf{T}}Ax_* - \frac{1}{2}y^{\mathsf{T}}Ay + b^{\mathsf{T}}x_* + b^{\mathsf{T}}y \\
&= \left|\begin{array}{l} x_*^{\mathsf{T}}Ay = (Ax_*)^{\mathsf{T}}y = b^{\mathsf{T}}y \quad (A = A^{\mathsf{T}}) \\ y^{\mathsf{T}}Ax_* = y^{\mathsf{T}}b = b^{\mathsf{T}}y \\ x_*^{\mathsf{T}}Ax_* = b^{\mathsf{T}}A^{-1}AA^{-1}b = b^{\mathsf{T}}A^{-1}b \\ b^{\mathsf{T}}x_* = b^{\mathsf{T}}A^{-1}b \end{array}\right| \\
&= -\frac{1}{2}b^{\mathsf{T}}A^{-1}b - \frac{1}{2}b^{\mathsf{T}}y - \frac{1}{2}b^{\mathsf{T}}y - \frac{1}{2}y^{\mathsf{T}}Ay + b^{\mathsf{T}}A^{-1}b + b^{\mathsf{T}}y \\
&= \frac{1}{2}\,b^{\mathsf{T}}A^{-1}b - \frac{1}{2}\,y^{\mathsf{T}}Ay.
\end{aligned}$$

All terms linear in y have cancelled, exactly as in one variable, leaving a constant plus a pure quadratic form in the fluctuation y. The constant comes out of the integral, the shift $x = x_* + y$ has unit Jacobian, and the y integral is Eq. (19):

$$\boxed{Z(b) = \int d^N x\; e^{-\frac{1}{2}x^{\mathsf{T}}Ax + b^{\mathsf{T}}x} = e^{\frac{1}{2}b^{\mathsf{T}}A^{-1}b}\int d^N y\; e^{-\frac{1}{2}y^{\mathsf{T}}Ay} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}\;\exp\Big[\frac{1}{2}\,b^{\mathsf{T}}A^{-1}b\Big]} \tag{21}$$

This is the central formula of these notes. Two objects built from A appear in it, with very different roles. The determinant sits in the prefactor and knows nothing about the sources. The inverse A^{-1} carries the entire dependence on b, and therefore, by differentiation, all the moments.

Moments in N variables. Exactly as in Eq. (15), each derivative $\partial/\partial b_i$ acting on $Z(b)$ brings down one factor of x_i from $e^{b^{\mathsf{T}}x} = e^{\sum_k b_k x_k}$, so the moments of the N-variable Gaussian are

$$\langle x_{i_1}\cdots x_{i_n}\rangle \equiv \frac{\int d^N x\; x_{i_1}\cdots x_{i_n}\, e^{-\frac{1}{2}x^{\mathsf{T}}Ax}}{\int d^N x\; e^{-\frac{1}{2}x^{\mathsf{T}}Ax}} = \frac{1}{Z(0)}\frac{\partial}{\partial b_{i_1}}\cdots\frac{\partial}{\partial b_{i_n}}Z(b)\Big|_{b=0} = \frac{\partial}{\partial b_{i_1}}\cdots\frac{\partial}{\partial b_{i_n}}e^{W(b)}\Big|_{b=0}, \tag{22}$$

where, writing $G \equiv A^{-1}$ for brevity,

$$W(b) \equiv \frac{1}{2}\,b^{\mathsf{T}}Gb = \frac{1}{2}\sum_{k,l} b_k G_{kl} b_l.$$

Everything follows from the derivatives of W, which is a quadratic polynomial in b and therefore has only two nonvanishing derivatives:

$$W_j \equiv \frac{\partial W}{\partial b_j} = \frac{1}{2}\sum_{k,l}(\delta_{kj}G_{kl}b_l + b_k G_{kl}\delta_{lj}) = \frac{1}{2}\sum_l G_{jl}b_l + \frac{1}{2}\sum_k b_k G_{kj} = \sum_l G_{jl}b_l,$$

$$W_{ij} \equiv \frac{\partial^2 W}{\partial b_i \partial b_j} = G_{ij}, \qquad \frac{\partial^3 W}{\partial b_i\, \partial b_j\, \partial b_k} = 0, \tag{23}$$

where $G_{kj} = G_{jk}$ was used. Two facts about the point $b = 0$ will be used repeatedly: $W(0) = 0$ so $e^{W(0)} = 1$, and $W_j(0) = 0$ because W_j is linear in b.

Two-point function. One derivative gives $\partial_j e^W = W_j e^W$. A second gives

$$\frac{\partial^2}{\partial b_i \partial b_j} e^W = \frac{\partial}{\partial b_i}(W_j e^W) = (W_{ij} + W_i W_j) e^W \xrightarrow{b=0} W_{ij} = G_{ij},$$

since $W_i W_j$ vanishes at $b = 0$ and $e^{W(0)} = 1$. Hence

$$\boxed{\langle x_i x_j \rangle = (A^{-1})_{ij}} \tag{24}$$

The two-point function is the inverse of the matrix in the quadratic form.

Four-point function. Continue differentiating:

$$\frac{\partial^3}{\partial b_k \partial b_i \partial b_j} e^W = \frac{\partial}{\partial b_k}\Big[(W_{ij} + W_i W_j) e^W\Big] = \Big(W_{ki} W_j + W_i W_{kj} + W_{ij} W_k + W_i W_j W_k\Big) e^W,$$

$$\frac{\partial^4}{\partial b_l \partial b_k \partial b_i \partial b_j} e^W = \Big(W_{ki} W_{lj} + W_{li} W_{kj} + W_{ij} W_{lk}\Big) e^W + \Big(\text{terms with at least one } W_\bullet\Big) e^W.$$

The terms in the second bracket come from differentiating $W_i W_j W_k$ once (leaving two first derivatives) or from letting ∂_l act on e^W (producing W_l), and every one of them contains a factor $W_\bullet$ that vanishes at $b = 0$. What survives is

$$\boxed{\langle x_i x_j x_k x_l \rangle = G_{ij} G_{kl} + G_{ik} G_{jl} + G_{il} G_{jk}, \qquad G = A^{-1}} \tag{25}$$

a sum over the three ways of pairing the four indices, each pair contributing one two-point function $G = \langle xx \rangle$.

Wick's theorem. The same mechanism works to all orders: each derivative either produces a new $W_\bullet$ or completes a pair by turning an existing $W_\bullet$ into a $W_{\bullet\bullet} = G_{\bullet\bullet}$, and at $b = 0$ only terms with no unpaired $W_\bullet$ left survive. Hence

$$\boxed{\langle x_{i_1} x_{i_2} \cdots x_{i_{2k}} \rangle = \sum_{\text{ways of pairing } i_1,\ldots,i_{2k}} \; \prod_{\text{pairs } (p,q)} (A^{-1})_{i_p i_q}, \qquad \langle x_{i_1} \cdots x_{i_{2k+1}} \rangle = 0.} \tag{26}$$

The oscillatory version. For the quantum-mechanical sign of the exponent the same steps go through with Eq. (13) used mode by mode. With $y = Ox$ and $\beta \equiv Ob$,

$$\int d^N x\, e^{\frac{i}{2} x^{\mathsf T} A x + i b^{\mathsf T} x} = \left| \begin{array}{l} y = Ox,\ d^N x = d^N y \\ x^{\mathsf T} A x = \sum_n a_n y_n^2 \\ b^{\mathsf T} x = b^{\mathsf T} O^{\mathsf T} y = \beta^{\mathsf T} y \end{array} \right| = \prod_{n=1}^{N} \int dy_n\, e^{\frac{i}{2} a_n y_n^2 + i \beta_n y_n} = \prod_{n=1}^{N} \sqrt{\frac{2\pi}{-i a_n}}\; e^{-i\beta_n^2/2a_n}$$

$$= \prod_{n=1}^{N} \sqrt{\frac{2\pi}{-i a_n}}\; \exp\Big[-\frac{i}{2}\, \beta^{\mathsf T} D^{-1} \beta\Big] = \left| \begin{array}{l} \beta^{\mathsf T} D^{-1} \beta = b^{\mathsf T} O^{\mathsf T} D^{-1} O b = b^{\mathsf T} A^{-1} b \\ \text{since } A^{-1} = (O^{\mathsf T} D O)^{-1} = O^{\mathsf T} D^{-1} O \end{array} \right|$$

so that, for real symmetric A with all $a_n \neq 0$,

$$\boxed{\int d^N x\, e^{\frac{i}{2} x^{\mathsf T} A x + i b^{\mathsf T} x} = \prod_{n=1}^{N} \sqrt{\frac{2\pi}{-i a_n}}\; \exp\Big[-\frac{i}{2}\, b^{\mathsf T} A^{-1} b\Big].} \tag{27}$$

Correspondingly the moments, defined with the weight $e^{\frac{i}{2} x^{\mathsf T} A x}$, are given by Wick's theorem with $G_{ij} = -i(A^{-1})_{ij}$ in place of $(A^{-1})_{ij}$, since each pair of derivatives $\frac{\partial}{\partial(ib_i)} \frac{\partial}{\partial(ib_j)}$ of $e^{-\frac{i}{2} b^{\mathsf T} A^{-1} b}$ at $b = 0$ gives $-i(A^{-1})_{ij}$.

4 Infinitely many variables: the free field

4.1 The dictionary and the integral to be done

To pass from N variables to a field, just read the previous section with the replacements

N variables		field theory
x_i, b_i, $\sum_i$, $\partial/\partial b_i$	$\longrightarrow$	$\phi(x)$, $J(x)$, $\int d^4x$, $\delta/\delta J(x)$
A_{ij}	$\longrightarrow$	a differential operator, here $-(-\Box + m^2)$
$(A^{-1})_{ij}$	$\longrightarrow$	its Green's function $G(x,y)$, with $(-\Box_x + m^2)G(x,y) = \delta^4(x-y)$

The integral to be done is $Z[J]$ of Eq. (1) for the free theory, with the exponent (2):

$$Z[J] = \int \mathcal{D}\phi \, \exp\Big[\, i \int d^4x \Big(-\frac{1}{2}\, \phi \, (-\Box + m^2 - i\epsilon)\, \phi + J\phi \Big) \Big]. \tag{28}$$

The $-i\epsilon$ has been included from the start. In lecture it arose from the replacement $t \to (1-i\epsilon)t$ that was needed to project the path integral onto the vacuum state. We will see in a moment that it is also exactly the damping of Method 1 that makes the oscillatory Gaussian converge. The result, to be derived below, is

$$\boxed{Z[J] = Z[0] \, \exp\Big[\frac{i}{2}\int d^4x \, d^4y \, J(x)\Delta(x-y)J(y)\Big], \quad \Delta(x-y) = \int \frac{d^4k}{(2\pi)^4} \, \frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon},} \tag{29}$$

with $Z[0] \propto [\det(-\Box + m^2 - i\epsilon)]^{-1/2}$ a number independent of J. This is Eq. (27) with $A = -(-\Box + m^2 - i\epsilon)$ and $b = J$: the exponent $-\frac{i}{2} b^{\mathsf{T}} A^{-1} b$ becomes $+\frac{i}{2}\int J(-\Box + m^2 - i\epsilon)^{-1} J$.

4.2 Doing the integral: Fourier transform

The way to diagonalise the operator $-\Box + m^2$ is to go to momentum space, since it is translation invariant and plane waves are its eigenfunctions. Fix conventions:

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} \, e^{ikx} \, \tilde{\phi}(k), \qquad \tilde{\phi}(k) = \int d^4x \, e^{-ikx} \, \phi(x), \qquad kx \equiv k_\mu x^\mu = -k^0 t + \vec{k}\cdot\vec{x}, \tag{30}$$

and the same for J. Because $\phi(x)$ is real, $\tilde{\phi}(-k) = \tilde{\phi}(k)^*$. Three facts are needed.
(i) Plane waves are eigenfunctions. Since $\partial_\mu e^{ikx} = ik_\mu e^{ikx}$,

$$\Box \, e^{ikx} = \partial_\mu \partial^\mu e^{ikx} = (ik_\mu)(ik^\mu)\, e^{ikx} = -k^2 \, e^{ikx}, \qquad k^2 \equiv k_\mu k^\mu = -(k^0)^2 + \vec{k}^{\,2},$$

so $(-\Box + m^2)\, e^{ikx} = (k^2 + m^2)\, e^{ikx}$, and therefore $(-\Box + m^2)\phi(x) = \int \frac{d^4k}{(2\pi)^4}(k^2+m^2) e^{ikx}\tilde{\phi}(k)$.
(ii) Orthogonality. $\int d^4x \, e^{i(k+k')x} = (2\pi)^4 \delta^4(k+k')$.
(iii) Products integrate mode by mode. For any two functions,

$$\int d^4x \, f(x) g(x) = \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \, \tilde{f}(k)\tilde{g}(k') \int d^4x \, e^{i(k+k')x} = \int \frac{d^4k}{(2\pi)^4} \, \tilde{f}(k)\, \tilde{g}(-k).$$

Applying (i) and (iii) to the two terms of the exponent in Eq. (28), and symmetrising the source term under $k \to -k$,

$$S + \int J\phi = \int \frac{d^4k}{(2\pi)^4} \Big[-\frac{1}{2}\, \tilde{\phi}(-k)\, (k^2 + m^2 - i\epsilon)\, \tilde{\phi}(k) + \frac{1}{2}\tilde{J}(-k)\tilde{\phi}(k) + \frac{1}{2}\tilde{J}(k)\tilde{\phi}(-k) \Big]. \tag{31}$$

Each momentum mode now talks only to itself and to its partner at $-k$: the operator has been diagonalised, with eigenvalue $k^2 + m^2 - i\epsilon$ on the mode k. Comparing with Eq. (27), the "a_n"

of the mode k is $-(k^2+m^2-i\epsilon)$, whose imaginary part is $+\epsilon>0$. This is precisely the sign that Method 1 required for convergence: the $i\epsilon$ that projected onto the vacuum in lecture is also the $i\epsilon$ that makes the Gaussian well defined.

Now complete the square mode by mode. Write $K(k)\equiv k^2+m^2-i\epsilon$, which is even in k. The stationary point of the exponent is at $K\tilde\phi=\tilde J$, so we shift

$$\tilde\phi(k)=\frac{\tilde J(k)}{K(k)}+\tilde\chi(k)$$

and expand the integrand of Eq. (31):

$$\begin{aligned}&-\frac{1}{2}K\Big(\frac{\tilde J(-k)}{K}+\tilde\chi(-k)\Big)\Big(\frac{\tilde J(k)}{K}+\tilde\chi(k)\Big)+\frac{1}{2}\tilde J(-k)\Big(\frac{\tilde J(k)}{K}+\tilde\chi(k)\Big)+\frac{1}{2}\tilde J(k)\Big(\frac{\tilde J(-k)}{K}+\tilde\chi(-k)\Big)\\&=-\frac{1}{2}K(k)\,\tilde\chi(-k)\tilde\chi(k)+\frac{\tilde J(k)\tilde J(-k)}{2K(k)}.\end{aligned}$$

Hence

$$S+\int J\phi=\int\frac{d^4k}{(2\pi)^4}\Big[-\frac{1}{2}\,\tilde\chi(-k)\,(k^2+m^2-i\epsilon)\,\tilde\chi(k)\Big]+\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{\tilde J(k)\tilde J(-k)}{k^2+m^2-i\epsilon}.\qquad(32)$$

The first term is the free action of the shifted field χ, and the second is a number depending only on J. A shift of the integration variable by a fixed function does not change the measure, $\mathcal{D}\phi=\mathcal{D}\chi$, so

$$Z[J]=\underbrace{\int\mathcal{D}\chi\;e^{\,iS[\chi]}}_{=Z[0]}\;\exp\Big[\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{\tilde J(k)\tilde J(-k)}{k^2+m^2-i\epsilon}\Big].$$

It remains to write the exponent in position space. Inserting the definition of $\tilde J$ from Eq. (30),

$$\begin{aligned}\int\frac{d^4k}{(2\pi)^4}\,\frac{\tilde J(k)\tilde J(-k)}{K(k)}&=\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{K(k)}\int d^4x\,e^{-ikx}J(x)\int d^4y\,e^{iky}J(y)\\&=\int d^4x\,d^4y\;J(x)J(y)\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{K(k)}=\int d^4x\,d^4y\;J(x)\,\Delta(x-y)\,J(y),\end{aligned}$$

which is Eq. (29). The function Δ is the inverse of the quadratic form; acting with the operator on the integrand,

$$(-\Box_x+m^2)\,\Delta(x-y)=\int\frac{d^4k}{(2\pi)^4}\,\frac{(k^2+m^2)\,e^{ik(x-y)}}{k^2+m^2-i\epsilon}\;\xrightarrow{\epsilon\to0}\;\int\frac{d^4k}{(2\pi)^4}\,e^{ik(x-y)}=\delta^4(x-y).$$

4.3 Correlation functions

Now apply the second half of Eq. (1). The functional derivative is defined by $\delta J(y)/\delta J(x)=\delta^4(x-y)$. Write $W[J]\equiv\frac{i}{2}\int d^4x\,d^4y\,J(x)\Delta(x-y)J(y)$, so that $Z[J]=Z[0]e^{W[J]}$. One derivative:

$$\begin{aligned}\frac{\delta W}{\delta J(x_1)}&=\frac{i}{2}\int d^4x\,d^4y\Big[\delta^4(x-x_1)\Delta(x-y)J(y)+J(x)\Delta(x-y)\delta^4(y-x_1)\Big]\\&=\frac{i}{2}\int d^4y\,\Delta(x_1-y)J(y)+\frac{i}{2}\int d^4x\,J(x)\Delta(x-x_1)=i\int d^4y\;\Delta(x_1-y)\,J(y),\end{aligned}$$

where the two terms were combined using $\Delta(x-x_1)=\Delta(x_1-x)$. This is linear in J and vanishes at $J=0$. A second derivative of W is a constant,

$$\frac{\delta^2W}{\delta J(x_2)\,\delta J(x_1)}=i\,\Delta(x_1-x_2),$$

and third derivatives vanish. In particular

$$\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)}\Big|_{J=0} = Z[0]\; i\,\Delta(x_1 - x_2),$$

and dividing by $i^2 Z[0]$ as Eq. (1) instructs,

$$\boxed{\langle 0|\,\mathrm{T}\{\phi(x_1)\phi(x_2)\}\,|0\rangle = \frac{1}{i}\,\Delta(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4}\,\frac{-i\,e^{ik(x_1-x_2)}}{k^2 + m^2 - i\epsilon}.} \tag{33}$$

Compare with what was found in lecture for the Feynman propagator,

$$G_F(x_1, x_2) = \langle 0|\,\mathrm{T}\{\phi(x_1)\phi(x_2)\}\,|0\rangle = \int \frac{d^4k}{(2\pi)^4}\,\frac{-i\,e^{ik(x_1-x_2)}}{k^2 + m^2 - i\epsilon},$$
$$Z[J] = Z[0]\,\exp\Big(-\frac{1}{2}\int d^4x\, d^4y\, J(x)G_F(x,y)J(y)\Big).$$

The two-point functions agree, and so do the generating functionals: $\Delta = iG_F$, hence $\frac{i}{2}\int J\Delta J = \frac{i}{2}\int J(iG_F)J = -\frac{1}{2}\int JG_F J$.