Section 4: Recap and Symmetries

Wednesday, September 30th 2026

Srednicki, Ch. 8–9, 22–24; Schwartz, Ch. 7, §3.3, §14.7–14.8

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1 The story so far

1.1 The plan

Before pushing on, this section takes a step back. Four weeks in, we have a lot of machinery: Lagrangians, mode expansions, propagators, path integrals, Gaussian integrals, Noether's theorem. It is easy to lose track of what it is all *for*. So we first recall, with as little formalism as possible, what problem quantum field theory solves and what the field is. Then we collect the results of the lectures on one page. The rest of the section is about symmetries, where the lectures are heading: what a symmetry is, what it conserves, how the conserved charge acts on the states, and how a symmetry tells us in advance which of the quantities we compute must vanish.

1.2 Why fields: particles come and go

Ordinary quantum mechanics describes a *fixed* set of particles. The Hilbert space of one particle is wavefunctions $\psi(\vec{x})$, of two particles $\psi(\vec{x}_1, \vec{x}_2)$, and so on; the number of arguments is fixed once and for all. Nature does not work like that: an excited atom emits a photon that was not there before, a photon of enough energy turns into an electron and a positron, a neutron decays into three particles. Whenever energies are comparable to mc^2, particles are created and destroyed. We need a Hilbert space in which states of $N = 0, 1, 2, \ldots$ particles live together, and operators that move between them.

We already know a quantum system that does exactly this: the harmonic oscillator. It has one ladder of states $|n\rangle$, and the ladder operators climb up and down it,

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle, \qquad a|n\rangle = \sqrt{n}\,|n-1\rangle, \qquad H = \omega(a^\dagger a + \frac{1}{2}). \tag{1}$$

Usually we read n as "the n-th excited state of a single object". Try to forget that reading and look only at the formulas: there is an integer n that can go up and down, an operator $a^\dagger$ that

raises it by one, an operator a that lowers it by one, and a total energy equal to n times ω (plus a constant). These are exactly the properties of a collection of n identical particles of energy ω: $a^\dagger$ creates one, a destroys one, energies add up, and nothing distinguishes one particle from another (there is only the number n, no labels). Reading the excitation number as a particle number is the idea behind *second quantisation*. Nothing has been added to the quantum mechanics of the oscillator; we have only changed what we call its states. The oscillator is already a theory of particles that can be created and annihilated. Its only shortcoming is that all its particles are the same: one energy ω, no momentum, and no notion of *where* they are.

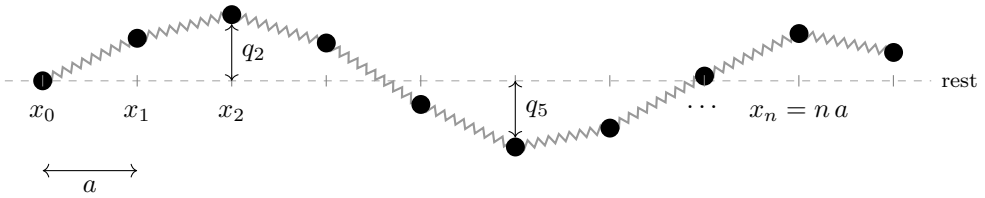
1.3 What the field is

Real particles carry a momentum $\vec{k}$ and an energy $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$, and they are created and destroyed *somewhere*. So we want quantum oscillators attached to the points of space. Take N of them for now, one at each point $x_n = n\,a$ of a line, and couple each to its two neighbours:

$$H = \sum_n \Big[\frac{\hat{p}_n^{\,2}}{2M} + \frac{1}{2}\mu\,\hat{q}_n^{\,2} + \frac{1}{2}\kappa\,(\hat{q}_{n+1} - \hat{q}_n)^2 \Big], \qquad [\hat{q}_n, \hat{p}_{n'}] = i\,\delta_{nn'}. \tag{2}$$

Each $\hat{q}_n$ is the coordinate operator of the oscillator at x_n; a state is a wavefunction $\psi(q_1, \ldots, q_N)$ of all N coordinates at once, and $\hat{p}_n = -i\,\partial/\partial q_n$.

Why couple them? Without the κ term the N oscillators would be independent: a quantum put into the oscillator at x_n would stay there forever, nothing could travel from one point to another, and no particle could *move* through space. The coupling lets a disturbance at x_n pull on $x_{n\pm1}$ and so propagate. It turns a collection of oscillators into a *medium*, and it is the discrete version of the $(\nabla\phi)^2$ term in $\mathcal{L}$. We couple only nearest neighbours because physics is local: what happens at x should depend on the field at x and in its immediate neighbourhood, i.e. on ϕ and its derivatives, not on the field far away.



Now let the oscillators become dense, $a \to 0$. The label n becomes a continuous label x, the coordinate operator of the oscillator at x_n becomes an operator $\phi(x)$ attached to the point x, and the wavefunction $\psi(q_1, \ldots, q_N)$ becomes a functional $\Psi[\phi]$, an amplitude for each possible shape $\phi(x)$ of the whole system (the wave functional of the lecture notes):

$$\hat{q}_n(t) \;\longrightarrow\; \phi(t,x), \qquad [\hat{q}_n, \hat{p}_{n'}] = i\delta_{nn'} \;\longrightarrow\; [\phi(t,x), \Pi(t,x')] = i\,\delta(x - x'). \tag{3}$$

This is the field: $\phi(x)$ is the coordinate operator $\hat{q}$ of the oscillator sitting at x. The point x is not the position of anything; it is the "name" of an oscillator. And $\phi(x)$ is an operator in exactly the sense that $\hat{q}$ is one: it has no value, only a distribution of values in a given state. Since we work in the Heisenberg picture, where operators depend on time, the field carries a time label as well, $\phi(t,x)$.

The ladder operators of the medium. Without the coupling each oscillator has its own ladder $a_n^\dagger, a_n$ and its own quanta, sitting at x_n. With the coupling, "oscillator n excited" is no longer a state of definite energy: the κ term contains $\hat{q}_n\hat{q}_{n+1}$, which moves the excitation to

the neighbour and back. So we must diagonalize H, i.e. find the normal coordinates. Here we can guess them. Every site of the chain looks the same (same M, μ, κ everywhere), so shifting the whole chain by one site is a symmetry of H, and the normal coordinates must be the combinations that only pick up a phase under $n \to n+1$: the plane waves. Define

$$\tilde{q}_k = \frac{1}{\sqrt{N}} \sum_n e^{-ikx_n}\, \hat{q}_n, \qquad \tilde{p}_k = \frac{1}{\sqrt{N}} \sum_n e^{-ikx_n}\, \hat{p}_n, \qquad k = \frac{2\pi j}{Na},\ j = 0, \ldots, N-1, \tag{4}$$

and use $\frac{1}{N}\sum_n e^{i(k-k')x_n} = \delta_{kk'}$. The three terms of H become

$$\sum_n \hat{p}_n^2 = \sum_k |\tilde{p}_k|^2, \qquad \sum_n \hat{q}_n^2 = \sum_k |\tilde{q}_k|^2,$$
$$\sum_n (\hat{q}_{n+1} - \hat{q}_n)^2 = \sum_k |e^{ika} - 1|^2\, |\tilde{q}_k|^2 = \sum_k 2(1 - \cos ka)\, |\tilde{q}_k|^2, \tag{5}$$

where in the last one the shift $n \to n+1$ produced the phase e^{ika}, and the cross terms between different k dropped out because of the $\delta_{kk'}$. Hence

$$H = \sum_k \Big[\frac{|\tilde{p}_k|^2}{2M} + \frac{1}{2} M \omega_k^2\, |\tilde{q}_k|^2 \Big], \qquad \omega_k^2 = \frac{\mu}{M} + \frac{2\kappa}{M}(1 - \cos ka) \xrightarrow{ka \ll 1} \frac{\mu}{M} + \frac{\kappa a^2}{M}\, k^2. \tag{6}$$

No term couples different k's: the plane-wave modes are *independent* oscillators, one for each k, with frequency ω_k. Each mode has its own ladder, a_k and $a_k^\dagger$, built from $\tilde{q}_k$ and $\tilde{p}_k$ exactly as for one oscillator, and

$$H = \sum_k \omega_k \Big(a_k^\dagger a_k + \frac{1}{2} \Big). \tag{7}$$

Now read the quanta as particles, as above: $|0\rangle$ is every mode in its ground state, $a_k^\dagger |0\rangle = |k\rangle$ is one particle of momentum k and energy ω_k, and products of $a^\dagger$'s build the Fock space. The particles of the medium are quanta of plane waves, not of individual oscillators; that is why they carry a definite momentum.

What the field operator does. Now go back from the modes to the oscillator at x_n by inverting the Fourier transform, $\hat{q}_n = \frac{1}{\sqrt{N}} \sum_k e^{ikx_n} \tilde{q}_k$. Each normal coordinate is, as for any oscillator, "create one quantum plus destroy one": $\tilde{q}_k = (a_k + a_{-k}^\dagger)/\sqrt{2M\omega_k}$ (the $-k$ appears because $\hat{q}_n$ is hermitian while $\tilde{q}_k$ is not). Inserting, and relabelling $k \to -k$ in the $a^\dagger$ term,

$$\hat{q}_n = \frac{1}{\sqrt{N}} \sum_k \frac{1}{\sqrt{2M\omega_k}} \Big[a_k\, e^{ikx_n} + a_k^\dagger\, e^{-ikx_n} \Big]. \tag{8}$$

So the coordinate of the oscillator at x_n creates or destroys one quantum in *every* mode, with the phase $e^{\mp ikx_n}$ of that mode at x_n. In the continuum limit $\sum_k / \sqrt{N}$ becomes an integral over $\vec{k}$, M becomes the normalization of the field, and the Heisenberg time dependence $a_k(t) = a_k e^{-i\omega_k t}$ supplies the $-\omega t$ in the exponent. The result is the mode expansion from lecture,

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\, 2\omega(\vec{k})} \Big[a(\vec{k})\, e^{ik_\mu x^\mu} + a^\dagger(\vec{k})\, e^{-ik_\mu x^\mu} \Big], \qquad k_\mu x^\mu = -\omega t + \vec{k} \cdot \vec{x}. \tag{9}$$

So $\phi(x)$ adds or removes *one* particle, in a superposition of all momenta with phases chosen so that the particle sits at $\vec{x}$. On the vacuum only the $a^\dagger$ half acts,

$$\phi(t, \vec{x})|0\rangle = \int \frac{d^3k}{(2\pi)^3\, 2\omega(\vec{k})}\, e^{i\omega t - i\vec{k}\cdot\vec{x}}\, |\vec{k}\rangle = \text{ one particle at } \vec{x}. \tag{10}$$

1.4 Correlation functions and time ordering

Reading $\langle 0|\,\phi(x)\phi(y)\,|0\rangle$. Operators act on what stands to their right, so read the expression from right to left, remembering that ϕ is the ladder operator of a point. $\phi(y)|0\rangle$ is the state "one particle at y", created out of the vacuum. $\langle 0|\phi(x)$ is the hermitian conjugate of $\phi(x)|0\rangle$, the bra of "one particle at x". So

$$G(x,y) = \langle 0|\,\phi(x)\phi(y)\,|0\rangle = \langle\,\text{particle at } x\,|\,\text{particle at } y\,\rangle = \int \frac{d^3k}{(2\pi)^3\,2\omega(\vec{k})}\,e^{ik_\mu(x-y)^\mu}, \quad (11)$$

the overlap between the state in which a particle has been created at y and the state in which one is annihilated at x. Since ϕ is a Heisenberg operator, the times are built in, so this is the *amplitude* for a particle created at y to be annihilated at x, with the free evolution in between[1]. Note that it is an amplitude, not a probability (the states $\phi(x)|0\rangle$ are not normalisable, so $|G|^2$ is a probability *density* at best).

Note that $G(x,y)$ is not zero when x and y are spacelike separated: the integral in Eq. (11) does not vanish there. So there is a nonzero amplitude for a particle created at y to be annihilated at x, even though no signal can travel between the two points. Does this contradict relativity? No. Relativity forbids something more specific: a measurement at x must not influence a measurement at y when the two points are spacelike separated. In quantum mechanics two measurements do not influence each other when the operators commute, so the requirement is

$$[\phi(x),\phi(y)] = 0 \qquad \text{for spacelike } x-y, \quad (12)$$

and it does hold. The commutator of two fields is a number, not an operator, because it is built from $[a,a^\dagger]$'s; so it equals its own vacuum expectation value, $[\phi(x),\phi(y)] = G(x,y) - G(y,x)$. For spacelike separation a Lorentz transformation can exchange the two points, and G is Lorentz invariant, so $G(x,y) = G(y,x)$ and the commutator vanishes. The amplitude is nonzero, but it is the same in both directions.

Why time ordering. The reading above only makes sense if y is the earlier point: create first, detect later. If instead $x^0 < y^0$, the amplitude for "a particle goes from one point to the other" is $\langle 0|\,\phi(y)\phi(x)\,|0\rangle$, with the earlier operator on the right so that it acts first. The time-ordered product is the single function that tells the story in the right direction whichever point is earlier:

$$G_F(x,y) = \langle 0|\,\mathrm{T}\{\phi(x)\phi(y)\}\,|0\rangle = \begin{cases} \langle 0|\,\phi(x)\phi(y)\,|0\rangle & x^0 > y^0 \\ \langle 0|\,\phi(y)\phi(x)\,|0\rangle & x^0 < y^0 \end{cases} \qquad \text{"the earlier field acts first".} \quad (13)$$

This is the Feynman propagator. Two more reasons make it, and not $G(x,y)$, the object the whole course is built on. Its defining rule is frame-independent: for timelike separation every observer agrees which point is earlier, and for spacelike separation the two fields commute, so the order does not matter. And it is what the path integral produces: the path integral is assembled slice by slice in time, so operators inserted into it automatically come out in time order.

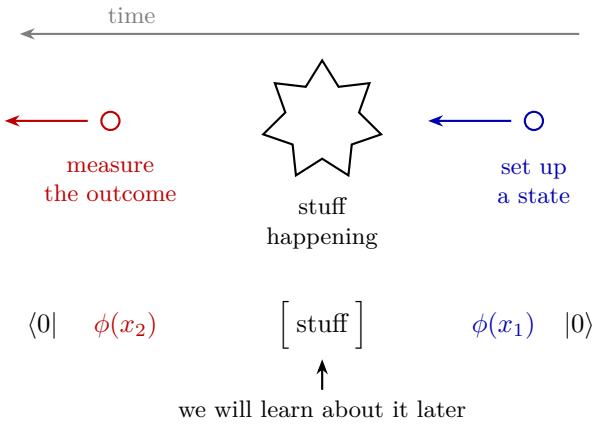
The same reading extends to n points. In

$$\langle 0|\,\mathrm{T}\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\}\,|0\rangle, \quad (14)$$

the earliest field acts first and creates a particle, later ones create or absorb, and the last one leaves the vacuum behind. It is the amplitude for particles to be created at some of the points, propagate, and be absorbed at the others.

[1] G is called the *Wightman function*

Why observables are built out of them. Because every experiment is one. An experiment prepares some particles, lets time pass, and detects some particles. In the language of the field: create quanta at some points, let the theory evolve, absorb quanta at other points, and ask for the amplitude. Read from right to left, in the order in which things happen:



The two-point function is the simplest experiment, one particle in and one particle out. Everything else we will measure is a longer version of the same question, with more fields inserted.

With this picture in mind, here is a recap of the main equations of the past weeks.

1.5 Recap

Classical field theory. The free real scalar, in signature $(-,+,+,+)$:

$$S[\phi] = \int d^4x\, \mathcal{L}, \qquad \mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \qquad \frac{\delta S}{\delta\phi} = 0 \implies (\Box - m^2)\phi = 0, \tag{15}$$

with $\Box = \partial_\mu\partial^\mu = -\partial_t^2 + \nabla^2$. Plane waves $e^{ik_\mu x^\mu}$ solve it when $k^0 = \pm\omega(\vec{k})$, $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$, and the general real solution is

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\Big[\tilde{\phi}(\vec{k})\, e^{ik_\mu x^\mu} + \tilde{\phi}^*(\vec{k})\, e^{-ik_\mu x^\mu}\Big], \qquad k^0 = \omega(\vec{k}). \tag{16}$$

The conjugate momentum and the Hamiltonian are

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}, \qquad H = \int d^3x\, \Big[\frac{1}{2}\Pi^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{2}m^2\phi^2\Big]. \tag{17}$$

The free quantum field. Canonical commutators, mode expansion, vacuum and particles:

$$[\phi(t,\vec{x}), \Pi(t,\vec{x}')] = i\delta^{(3)}(\vec{x}-\vec{x}'), \qquad \Pi = \dot{\phi}, \tag{18}$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\, 2\omega(\vec{k})}\Big[a(\vec{k})\, e^{ik_\mu x^\mu} + a^\dagger(\vec{k})\, e^{-ik_\mu x^\mu}\Big], \qquad [a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3\, 2\omega(\vec{k})\, \delta^{(3)}(\vec{k}-\vec{k}'), \tag{19}$$

$$a(\vec{k})|0\rangle = 0, \qquad |\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle \ \text{ has energy } \omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2} \text{ and momentum } \vec{k}, \tag{20}$$

$$\langle\vec{k}|\vec{k}'\rangle = (2\pi)^3\, 2\omega(\vec{k})\, \delta^{(3)}(\vec{k}-\vec{k}'), \tag{21}$$

$$H = \int d^3x\, \Big[\frac{1}{2}\Pi^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{2}m^2\phi^2\Big] = \int \frac{d^3k}{(2\pi)^3\, 2\omega(\vec{k})}\, \omega(\vec{k})\, a^\dagger(\vec{k})a(\vec{k}) + \text{const.} \tag{22}$$

The time-ordered two-point function is the Feynman propagator,

$$G_F(x,y) = \langle 0|\,\mathrm{T}\{\phi(x)\phi(y)\}\,|0\rangle = \int \frac{d^4p}{(2\pi)^4}\,\frac{-i}{p^2+m^2-i\epsilon}\,e^{ip_\mu(x-y)^\mu},$$
$$(\Box_x - m^2)\,G_F(x,y) = i\,\delta^{(4)}(x-y). \tag{23}$$

Path integrals. In quantum mechanics, slicing the time evolution into small steps and inserting complete sets of states gives

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q\; e^{\,iS[q]}, \qquad S[q] = \int_{t'}^{t''} dt\,\Big[\frac{1}{2}m\dot{q}^2 - V(q)\Big], \tag{24}$$

an integral over all paths from q' to q''. In field theory, with a source J coupled to the field,

$$Z[J] = \int \mathcal{D}\phi\; e^{\,i\left(S[\phi]+\int d^4x\,J\phi\right)},$$
$$\langle 0|\,\mathrm{T}\{\phi(x_1)\cdots\phi(x_n)\}\,|0\rangle = \frac{1}{Z[0]}\Big(-i\frac{\delta}{\delta J(x_1)}\Big)\cdots\Big(-i\frac{\delta}{\delta J(x_n)}\Big)Z[J]\Big|_{J=0}, \tag{25}$$

where the vacuum is selected by tilting the time contour, $t \to t(1-i\epsilon)$. For the free theory the integral is Gaussian, and completing the square gives

$$Z_0[J] = Z_0[0]\,\exp\Big[-\frac{1}{2}\int d^4x\,d^4y\;J(x)\,G_F(x,y)\,J(y)\Big], \tag{26}$$

whose fourth derivative is a sum over pairings with one propagator per pair:

$$\langle 0|\,\mathrm{T}\{\phi_1\phi_2\phi_3\phi_4\}\,|0\rangle = G_F(x_1,x_2)G_F(x_3,x_4) + G_F(x_1,x_3)G_F(x_2,x_4) + G_F(x_1,x_4)G_F(x_2,x_3). \tag{27}$$

Shifting the integration variable, $\phi \to \phi + \delta\phi$, changes nothing, and expanding to first order gives the Schwinger–Dyson equation,

$$\Big\langle 0\Big|\mathrm{T}'\Big\{\frac{\delta S}{\delta\phi(x)}\,\phi(x_1)\cdots\phi(x_n)\Big\}\Big|0\Big\rangle = i\sum_{j=1}^{n}\delta^{(4)}(x-x_j)\,\langle 0|\,\mathrm{T}\{\phi(x_1)\cdots\phi(x_{j-1})\phi(x_{j+1})\cdots\phi(x_n)\}\,|0\rangle, \tag{28}$$

where T' means the time derivatives in $\delta S/\delta\phi$ act after time ordering. Valid in any theory, free or interacting.

2 Symmetries

Throughout this chapter φ stands for any field of the theory: the real scalar, or ϕ and $\phi^\dagger$ for the complex one.

2.1 Noether's theorem and conserved charges

A symmetry is a change of the fields that leaves the action alone. Noether's theorem says that every *continuous* symmetry comes with a conserved current. The derivation is short, so we go through it again.

Take a transformation with a small constant parameter α, $\varphi \mapsto \varphi + \alpha\,\Delta\varphi$, under which the Lagrangian does not change. (For now assume $\mathcal{L}$ itself is invariant.) Expanding $\mathcal{L}(\varphi, \partial_\mu\varphi)$ to first order in α, with $\partial_\mu\varphi \mapsto \partial_\mu\varphi + \alpha\,\partial_\mu\Delta\varphi$, the requirement reads

$$0 = \delta\mathcal{L} = \alpha\Big[\frac{\partial\mathcal{L}}{\partial\varphi}\,\Delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\,\partial_\mu\Delta\varphi\Big]. \tag{29}$$

The same expansion appears in the derivation of the Euler–Lagrange equations, with $\delta\varphi = \alpha\Delta\varphi$, but the question asked is different. There $\delta\varphi$ was arbitrary and we asked for the action to be stationary at one particular φ, the solution. Here $\Delta\varphi$ is a specific function of the fields (e.g. $\Delta\phi = -i\phi$ for a phase rotation) and we ask that $\mathcal{L}$ not change for *every* φ, solution or not. Eq. (29) is a condition on $\mathcal{L}$, not on φ.

Now do the same thing with a parameter that varies from point to point, $\alpha(x)$, vanishing outside some bounded region. That is no longer a symmetry, so $\mathcal{L}$ changes. Let us see how. Expand $\mathcal{L}(\varphi, \partial_\mu\varphi)$ to first order; the derivative of the field now picks up a term where the derivative hits α, $\partial_\mu\varphi \to \partial_\mu\varphi + \alpha\,\partial_\mu\Delta\varphi + (\partial_\mu\alpha)\,\Delta\varphi$:

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\varphi}\,\alpha\Delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Big[\alpha\,\partial_\mu\Delta\varphi + (\partial_\mu\alpha)\,\Delta\varphi\Big] \\ &= \alpha\,\underbrace{\Big[\frac{\partial\mathcal{L}}{\partial\varphi}\Delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\Delta\varphi\Big]}_{=0\ \text{by Eq. (29)}} + \partial_\mu\alpha\ \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi}_{\equiv j^\mu}.\end{aligned} \tag{30}$$

The first bracket is the change of $\mathcal{L}$ under the transformation with *constant* α, Eq. (29), and vanishes because that is a symmetry. Wherever α is locally constant nothing happens; only the term where the derivative hit α survives. Its coefficient is what we call the current,

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\,\Delta\varphi \tag{31}$$

or, if several fields φ_a transform at once, $\delta\varphi_a = \alpha\Delta\varphi_a$,

$$j^\mu = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\,\Delta\varphi_a, \tag{32}$$

since each field contributes its own term to the expansion of $\mathcal{L}$. Now integrate Eq. (30) over spacetime to get the change of the action,

$$S[\varphi + \alpha(x)\Delta\varphi] = S[\varphi] + \int d^4x\ \partial_\mu\alpha(x)\,j^\mu(x) + O(\alpha^2) = S[\varphi] - \int d^4x\ \alpha(x)\,\partial_\mu j^\mu(x) + O(\alpha^2), \tag{33}$$

where in the last step we integrated by parts; the boundary term drops because α vanishes outside a bounded region. Now take φ to be a solution of the classical equations of motion. By definition of a solution, the action is stationary: $\delta S = 0$ for *every small* change of the fields that vanishes at infinity, and $\alpha(x)\Delta\varphi$ is such a change. So the integral in Eq. (33) must vanish for every function $\alpha(x)$ we care to choose, which is only possible if the thing multiplying it is zero everywhere:

$$\partial_\mu j^\mu = 0 \qquad \Longleftrightarrow \qquad \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0, \qquad \rho \equiv j^0. \tag{34}$$

That is Noether's theorem. Notice the logic: the symmetry (constant α) killed the first bracket in Eq. (30), and the equations of motion (arbitrary $\alpha(x)$) killed what was left.

The second form is the continuity equation of fluid mechanics or electromagnetism: ρ is a density of "stuff", $\vec{j}$ is how the stuff flows, and the amount in any region changes only by what flows through the walls. Stuff is never created or destroyed. The total amount of stuff is called the *charge*, and it is constant in time:

$$Q = \int d^3x\ \rho(t,\vec{x}) \qquad \text{obeys} \qquad \frac{dQ}{dt} = -\int d^3x\ \nabla\cdot\vec{j} = -\oint_{\text{walls}} \vec{j}\cdot d\vec{A} = 0 \tag{35}$$

when the walls are at infinity and the fields vanish there.

Example from lecture. The complex scalar $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$,

$$\mathcal{L} = -\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi, \qquad \phi \mapsto e^{i\alpha}\phi, \quad \phi^\dagger \mapsto e^{-i\alpha}\phi^\dagger, \tag{36}$$

has particles ($a^\dagger$) and antiparticles ($b^\dagger$) of equal mass. In lecture we found by inspection the conserved current and charge

$$j^\mu = -i(\phi^\dagger\partial^\mu\phi - \phi\,\partial^\mu\phi^\dagger), \qquad Q = \int d^3x\ j^0 = \int \frac{d^3k}{(2\pi)^3\,2\omega(\vec{k})}\Big[a^\dagger a - b^\dagger b\Big] = N_a - N_b. \tag{37}$$

Let us recover them from Noether's formula. Take the phase rotation in the form $\phi \mapsto e^{-i\alpha}\phi$, i.e. $\Delta\phi = -i\phi$, $\Delta\phi^\dagger = +i\phi^\dagger$. (The sign of α is a convention. With the lecture's $\phi \mapsto e^{i\alpha}\phi$, Eq. (31) gives $-j^\mu$; we take $e^{-i\alpha}$ so that the Noether current is exactly Eq. (37), with $Q = N_a - N_b$.) Then

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\Delta\phi^\dagger = (-\partial^\mu\phi^\dagger)(-i\phi) + (-\partial^\mu\phi)(i\phi^\dagger) = -i(\phi^\dagger\partial^\mu\phi - \phi\,\partial^\mu\phi^\dagger), \tag{38}$$

which is Eq. (37): $\rho = j^0$ is (particle density) − (antiparticle density) and $Q = N_a - N_b$. The "stuff" becomes electric charge once the current is coupled to the photon.

2.2 The charge is the generator of the symmetry

Noether's theorem is a classical statement: on solutions of the equations of motion, a number Q does not change in time. In the quantum theory j^μ and Q are operators, and Q does a second job that the classical theorem does not mention: it *performs* the symmetry transformation on the Hilbert space. Let us see this for the complex scalar, where everything can be checked with the mode expansion.

Q on the states. With $\phi \sim a + b^\dagger$ inserted into $Q = \int d^3x\, j^0$, the cross terms $a^\dagger b^\dagger$ and ab cancel and one finds

$$Q = \int \frac{d^3k}{(2\pi)^3\,2\omega(\vec{k})}\Big[a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\Big] = N_a - N_b. \tag{39}$$

So $Q|0\rangle = 0$, and on a Fock state Q counts particles minus antiparticles: every state built from creation operators is an eigenstate of Q with an integer eigenvalue. To see how Q acts on the ladder operators, use $[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3\,2\omega(\vec{k})\,\delta^{(3)}(\vec{k}-\vec{k}')$:

$$\begin{aligned}[N_a, a^\dagger(\vec{k}')] &= \int \frac{d^3k}{(2\pi)^3\,2\omega(\vec{k})}\ a^\dagger(\vec{k})\,[a(\vec{k}), a^\dagger(\vec{k}')] = \int \frac{d^3k}{(2\pi)^3\,2\omega(\vec{k})}\ (2\pi)^3\,2\omega(\vec{k})\,\delta^{(3)}(\vec{k}-\vec{k}')\,a^\dagger(\vec{k}) \\ &= a^\dagger(\vec{k}'),\end{aligned} \tag{40}$$

and in the same way $[N_a, a(\vec{k}')] = -a(\vec{k}')$, with the same identities for b and N_b. Hence

$$[Q, a^\dagger(\vec{k})] = +a^\dagger(\vec{k}), \qquad [Q, b^\dagger(\vec{k})] = -b^\dagger(\vec{k}), \qquad [Q, a(\vec{k})] = -a(\vec{k}), \qquad [Q, b(\vec{k})] = +b(\vec{k})\,: \tag{41}$$

$a^\dagger$ raises the charge by one, $b^\dagger$ lowers it, and the annihilation operators do the opposite.

Q on the field. Since ϕ contains a and $b^\dagger$, both of which lower Q by one,

$$[Q,\phi] = -\phi, \qquad [Q,\phi^\dagger] = +\phi^\dagger. \tag{42}$$

Read this as: *ϕ removes one unit of charge from any state it acts on*, by annihilating a particle or by creating an antiparticle, and $\phi^\dagger$ adds one. We write q for the charge of an operator, defined by $[Q,\varphi] = -q\varphi$, so $q_\phi = +1$, $q_{\phi^\dagger} = -1$, and $q = 0$ for anything neutral such as $\phi^\dagger\phi$ or j^μ.

Now exponentiate. Using $[Q,\phi] = -\phi$ repeatedly,

$$e^{i\alpha Q}\,\phi\,e^{-i\alpha Q} = \phi + i\alpha[Q,\phi] + \frac{(i\alpha)^2}{2}[Q,[Q,\phi]] + \cdots = \phi\Big(1 - i\alpha + \frac{(-i\alpha)^2}{2} + \dots\Big) = e^{-i\alpha}\phi. \tag{43}$$

We recover the phase rotation we started from, in the convention chosen above. So the unitary operator

$$U(\alpha) = e^{i\alpha Q} \tag{44}$$

performs the symmetry on the Hilbert space: it rotates the phase of the field, leaves the vacuum alone ($U|0\rangle = |0\rangle$ because $Q|0\rangle = 0$), and multiplies a state of charge q by $e^{i\alpha q}$. The same quantity that Noether hands us as conserved is the generator of the transformation. The pattern is familiar from quantum mechanics: momentum generates translations, angular momentum generates rotations, the Hamiltonian generates time evolution, and here the charge generates phase rotations.

2.3 Selection rules

In quantum mechanics a symmetry tells you in advance that certain expectation values vanish. In a symmetric potential, $V(-x) = V(x)$, parity gives $\langle n|\hat{x}|n\rangle = 0$ for every energy eigenstate, without solving for a single wavefunction. Such statements are *selection rules*, and their proof is always the same two lines: insert the symmetry operator and its inverse, let one act on the states and the other on the operator, and compare. In field theory the quantities we compute are the correlation functions, and the same two lines tell us, before any path integral is done, which of them vanish and which are equal.

The sign flip. The simplest case is a real scalar with a Lagrangian even in ϕ and the symmetry $\phi \mapsto -\phi$. It has no parameter, hence no current and no charge. But like every symmetry it is performed on the Hilbert space by a unitary operator U, and all we need to know about U is what it does to the field and to the vacuum:

$$U\phi(x)\,U^{-1} = -\phi(x), \qquad U|0\rangle = |0\rangle. \tag{45}$$

Now insert $U^{-1}U$ on both sides of a correlator and between every two fields, and let the outer factors act on the vacuum. Each field is turned into $U\phi(x_j)U^{-1} = -\phi(x_j)$ and contributes one factor of -1,

$$\begin{aligned}\langle 0|\,\mathrm{T}\{\phi(x_1)\cdots\phi(x_n)\}\,|0\rangle &= \langle 0|U^{-1}\,\mathrm{T}\{(U\phi(x_1)U^{-1})(U\phi(x_2)U^{-1})\cdots(U\phi(x_n)U^{-1})\}\,U|0\rangle \\ &= (-1)^n\,\langle 0|\mathrm{T}\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle,\end{aligned} \tag{46}$$

so every correlator with an odd number of fields vanishes.

The continuous case. For a continuous symmetry $U = e^{i\alpha Q}$. We have $Q|0\rangle = 0$, so $U|0\rangle = |0\rangle$, and $U\varphi_j(x_j)U^{-1} = e^{-i\alpha q_j}\varphi_j(x_j)$ for a field φ_j of charge q_j (the index j labels which field, ϕ or $\phi^\dagger$, sits at the point x_j). The same two lines give

$$\begin{aligned}\langle 0|\,\mathrm{T}\{\varphi_1(x_1)\cdots\varphi_n(x_n)\}\,|0\rangle &= \langle 0|U^{-1}\,\mathrm{T}\{(U\varphi_1(x_1)U^{-1})\cdots(U\varphi_n(x_n)U^{-1})\}\,U|0\rangle \\ &= e^{-i\alpha\sum_j q_j}\,\langle 0|\,\mathrm{T}\{\varphi_1(x_1)\cdots\varphi_n(x_n)\}\,|0\rangle \end{aligned} \tag{47}$$

for every α, hence

$$\boxed{\langle 0|\,\mathrm{T}\{\varphi_1(x_1)\cdots\varphi_n(x_n)\}\,|0\rangle = 0 \quad \text{unless} \quad \sum_j q_j = 0.} \tag{48}$$

Here the φ_j are whichever fields of the theory sit at the points x_j; for the complex scalar each φ_j is either ϕ or $\phi^\dagger$. The number q_j is the charge of that field, defined by $[Q,\varphi_j] = -q_j\varphi_j$: the amount by which φ_j changes the charge of any state it acts on. From Eq. (42), ϕ has $q=+1$ (it removes one unit of charge, by annihilating a particle or creating an antiparticle), $\phi^\dagger$ has $q=-1$ (it adds one unit), and a product such as $\phi^\dagger\phi$ or the current j^μ has $q=0$. Once the current is coupled to the photon, eQ is the electric charge, and the q_j are the electric charges of the fields in units of e.

Now read Eq. (48) from right to left, as an experiment. The rightmost fields act on the vacuum, which has charge zero, and set up a charged state; the leftmost fields must bring the charge back to zero, because $\langle 0|$ has charge zero too. If they do not, the amplitude vanishes. For the complex scalar:

$$\begin{aligned} \langle 0|\,\mathrm{T}\{\phi(x_2)\,\phi^\dagger(x_1)\}\,|0\rangle &\neq 0, & \langle 0|\,\mathrm{T}\{\phi(x_2)\,\phi(x_1)\}\,|0\rangle &= 0, \\ \langle 0|\,\mathrm{T}\{\phi(x_4)\phi(x_3)\,\phi^\dagger(x_2)\phi^\dagger(x_1)\}\,|0\rangle &\neq 0, & \langle 0|\,\mathrm{T}\{\phi(x_4)\phi(x_3)\phi(x_2)\,\phi^\dagger(x_1)\}\,|0\rangle &= 0. \end{aligned} \tag{49}$$

The first is a particle created at x_1 and removed at x_2 (charges $-1+1=0$). The second would create a particle and then remove an antiparticle ($+1+1\neq 0$). The third is two particles in, two particles out: particle–particle scattering. The fourth would be one particle turning into two ($+1+1+1-1\neq 0$), which charge conservation forbids whatever the interactions. Suppose the theory also contains a field χ of charge 2. Then $\langle 0|\,\mathrm{T}\{\chi(x_3)\,\phi^\dagger(x_2)\phi^\dagger(x_1)\}\,|0\rangle$ has $2-1-1=0$ and is allowed: two ϕ-particles can turn into one χ-particle. The same counting decides which terms may appear in the Lagrangian: $\chi^\dagger\phi\phi$ has total charge zero and is an allowed interaction, ϕ^3 is not.

For the free complex scalar the first two statements can be checked from the mode expansion: $\phi \sim a + b^\dagger$, so $\phi\phi$ contains no $aa^\dagger$ or $bb^\dagger$ pair and its vacuum expectation value vanishes. But nothing in the argument used the free theory. The rule holds in any theory with the symmetry, to all orders in any coupling.

2.4 Several charges at once: N real scalars

The complex scalar had one conserved charge, and we got quite a lot out of it. What happens when a theory has *several* conserved charges?

Take N real scalar fields of the same mass,

$$\mathcal{L} = \sum_{i=1}^{N}\left(-\frac{1}{2}\,\partial_\mu\phi_i\,\partial^\mu\phi_i - \frac{1}{2}m^2\,\phi_i\phi_i\right) = -\frac{1}{2}\,\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{1}{2}m^2\,\vec{\phi}\cdot\vec{\phi}, \tag{50}$$

and think of $\vec{\phi} = (\phi_1, \ldots, \phi_N)$ as a vector in an internal N-dimensional space, one vector at each point x. Both terms in $\mathcal{L}$ are dot products of that vector with itself, and a dot product does not change when both factors are rotated by the same orthogonal matrix. So

$$\phi_i(x) \;\mapsto\; R_{ij}\,\phi_j(x), \qquad R^{\mathsf{T}}R = \mathbf{1}, \qquad R \text{ the same at every } x, \tag{51}$$

is a symmetry. The matrices with $R^{\mathsf{T}}R = \mathbf{1}$ form the group $O(N)$. Those reached continuously from $\mathbf{1}$ have $\det R = 1$ and form $SO(N)$, the rotations proper; Noether's theorem is about these. The rest, with $\det R = -1$, are reflections such as $\phi_1 \mapsto -\phi_1$, symmetries with no parameter to vary, of the kind treated in the previous subsection.

For $N = 2$ we recover the complex scalar from the lecture: with $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, the phase rotation $\phi \mapsto e^{-i\alpha}\phi$ used above is the rotation of the (ϕ_1, ϕ_2) plane by the angle α, $U(1) \simeq SO(2)$: one angle, one current, one charge.

Counting the symmetries. Near the identity write $R = \mathbf{1}+\theta$. Then $R^{\mathsf{T}}R = \mathbf{1}+\theta+\theta^{\mathsf{T}}+O(\theta^2)$, so $\theta^{\mathsf{T}} = -\theta$: a real antisymmetric matrix, with zeros on the diagonal and $N(N-1)/2$ independent entries θ_{kl}, $k < l$, above it. Each entry is the angle of a rotation in one coordinate plane. For $N = 3$,

$$\theta = \begin{pmatrix} 0 & \theta_{12} & \theta_{13} \\ -\theta_{12} & 0 & \theta_{23} \\ -\theta_{13} & -\theta_{23} & 0 \end{pmatrix}, \qquad \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \mapsto \begin{pmatrix} \phi_1 + \theta_{12}\phi_2 + \theta_{13}\phi_3 \\ \phi_2 - \theta_{12}\phi_1 + \theta_{23}\phi_3 \\ \phi_3 - \theta_{13}\phi_1 - \theta_{23}\phi_2 \end{pmatrix}, \tag{52}$$

so ϕ_1 is tilted towards ϕ_2 by the angle θ_{12} and towards ϕ_3 by the angle θ_{13}, i.e. rotated in the (ϕ_1, ϕ_2) and (ϕ_1, ϕ_3) planes. The third entry, θ_{23}, does not touch ϕ_1 at all: it rotates the (ϕ_2, ϕ_3) plane, in which ϕ_1 does not lie. A basis for the infinitesimal rotations is therefore the rotations in the individual planes,

$$\Delta_{kl}\phi_k = \phi_l, \qquad \Delta_{kl}\phi_l = -\phi_k, \qquad \Delta_{kl}\phi_i = 0 \;\text{ for } i \neq k, l, \qquad \text{i.e.} \qquad \Delta_{kl}\phi_i = \delta_{ik}\phi_l - \delta_{il}\phi_k, \tag{53}$$

oriented so that for $N = 2$, $\Delta_{12}\phi_1 = \phi_2$, $\Delta_{12}\phi_2 = -\phi_1$ is the transformation used above for the complex scalar: $\Delta\phi = (\Delta_{12}\phi_1 + i\Delta_{12}\phi_2)/\sqrt{2} = (\phi_2 - i\phi_1)/\sqrt{2} = -i\phi$.

One current per plane. Noether's formula Eq. (31) is linear in $\Delta\varphi$, so we apply it once per basis transformation. With $\partial\mathcal{L}/\partial(\partial_\mu\phi_i) = -\partial^\mu\phi_i$,

$$j^\mu_{kl} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\Delta_{kl}\phi_i = -\partial^\mu\phi_i(\delta_{ik}\phi_l - \delta_{il}\phi_k) = \phi_k\,\partial^\mu\phi_l - \phi_l\,\partial^\mu\phi_k. \tag{54}$$

The charges are, with $j^0 = -j_0$ and $\Pi_i = \dot{\phi}_i$,

$$Q_{kl} = \int d^3x\; j^0_{kl} = \int d^3x\; (\phi_l\,\Pi_k - \phi_k\,\Pi_l), \qquad k < l. \tag{55}$$

Each Q_{kl} has the form of an angular momentum, $x_k p_l - x_l p_k$ with $x \to \phi$ and $p \to \Pi$. So the theory has $N(N-1)/2$ conserved charges, one for each plane in which the field vector can rotate.

The charges do not commute. Take $N = 3$ and work out the commutator of two of the charges in Eq. (55) directly from the equal-time canonical commutators $[\phi_i(\vec{x}), \Pi_j(\vec{y})] = i\delta_{ij}\delta^{(3)}(\vec{x} - \vec{y})$, all other pairs commuting. Write

$$[Q_{12}, Q_{23}] = \int d^3x\, d^3y\; [\,\phi_2\Pi_1 - \phi_1\Pi_2\,,\; \phi_3\Pi_2 - \phi_2\Pi_3\,], \tag{56}$$

with the first factor at $\vec{x}$ and the second at $\vec{y}$. The only operators that fail to commute are ϕ_2 and Π_2, and each charge contains exactly one of the two. Expanding gives four terms. The two in which ϕ_2 meets ϕ_2, or Π_2 meets Π_2, vanish; the other two are

$$[\phi_2\Pi_1, \phi_3\Pi_2] = \Pi_1\phi_3[\phi_2(\vec{x}), \Pi_2(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\,\phi_3\Pi_1, \quad [\phi_1\Pi_2, \phi_2\Pi_3] = -i\delta^{(3)}(\vec{x}-\vec{y})\,\phi_1\Pi_3. \tag{57}$$

Doing the $\vec{y}$ integral with the delta function,

$$[Q_{12}, Q_{23}] = i\int d^3x\,\phi_3\Pi_1 - i\int d^3x\,\phi_1\Pi_3 = -i\int d^3x\,(\phi_1\Pi_3 - \phi_3\Pi_1) = -i\,Q_{31}. \tag{58}$$

The two charges do not commute, and their commutator is the third charge. For any N the same computation gives, for k, l, m all different and with $Q_{lk} \equiv -Q_{kl}$,

$$[Q_{kl}, Q_{lm}] = -i\,Q_{mk}, \qquad [Q_{kl}, Q_{mn}] = 0 \ \text{ if the planes share no axis,} \tag{59}$$

so two charges fail to commute exactly when their planes share an axis, and then their commutator is the charge of the plane spanned by the two remaining axes. The charges close among themselves under commutation. Behind this is the fact that the rotations themselves do not commute: just as $[Q, \phi] = -\phi$ made $e^{i\alpha Q}$ perform the phase rotation, the same canonical computation gives $[Q_{kl}, \phi_i] = -i\Delta_{kl}\phi_i$, so $e^{i\theta Q_{kl}}$ performs the rotation in the (k, l) plane, and operators performing non-commuting rotations cannot commute. A symmetry whose charges do not commute is called *nonabelian*.

The same in the language of generators. Rotations are more often written with *generators*: the real antisymmetric matrix θ of Eq. (52) is expanded in a fixed basis of real antisymmetric matrices T^a,

$$\theta = \theta^a T^a, \qquad a = 1, \ldots, \tfrac{N(N-1)}{2}, \qquad \text{so that} \qquad R = \mathbf{1} + \theta^a T^a + \cdots = e^{\theta^a T^a}, \tag{60}$$

with real parameters θ^a (the finite rotation is again obtained by compounding many small ones). The commutator of two antisymmetric matrices is antisymmetric, so it is a combination of the T^a again,

$$[T^a, T^b] = f^{abc}\,T^c, \tag{61}$$

with real *structure constants* f^{abc}, which encode how rotations compose. For $N = 3$ one can take $(T^a)_{ij} = \epsilon_{aij}$, i.e. T^3 is the rotation of the (ϕ_1, ϕ_2) plane, T^1 of the (ϕ_2, ϕ_3) plane and T^2 of the (ϕ_3, ϕ_1) plane (compare Eq. (52): $\theta^3 = \theta_{12}$, $\theta^1 = \theta_{23}$, $\theta^2 = \theta_{31}$), and a short computation gives $f^{abc} = -\epsilon^{abc}$. The transformation belonging to T^a is $\Delta_a\phi_i = (T^a)_{ij}\phi_j$, and its charge is, by Eq. (31),

$$Q^a = \int d^3x\,(-\partial^0\phi_i)(T^a)_{ij}\phi_j = \int d^3x\,\Pi_i\,(T^a)_{ij}\,\phi_j = \sum_{k<l}(T^a)_{kl}\,Q_{kl}, \tag{62}$$

a linear combination of the plane charges (for $N = 3$: $Q^3 = Q_{12}$, $Q^1 = Q_{23}$, $Q^2 = Q_{31}$). The commutator of two of them is the computation of Eq. (58) with all indices at once; only $[\phi_j, \Pi_k]$ and $[\Pi_i, \phi_l]$ contribute:

$$\begin{aligned}[Q^a, Q^b] &= \int d^3x\,d^3y\,(T^a)_{ij}(T^b)_{kl}\,[\Pi_i\phi_j,\ \Pi_k\phi_l] = i\int d^3x\,(T^a)_{ij}(T^b)_{kl}\big(\delta_{jk}\,\Pi_i\phi_l - \delta_{il}\,\Pi_k\phi_j\big) \\ &= i\int d^3x\,\Pi_i\big(T^aT^b - T^bT^a\big)_{il}\phi_l = i\,f^{abc}\int d^3x\,\Pi_i\,(T^c)_{il}\,\phi_l = i\,f^{abc}\,Q^c,\end{aligned} \tag{63}$$

i.e.

$$\boxed{[Q^a, Q^b] = i\, f^{abc}\, Q^c} \tag{64}$$

The charges obey the same algebra as the matrices that generate the rotations, with the same structure constants; the extra i comes from the canonical commutator. (Check for $N = 3$: $[Q^3, Q^1] = if^{312}Q^2 = -iQ^2$, which is Eq. (58).) Whatever group theory one knows about $SO(N)$ therefore applies verbatim to the conserved charges. For N *complex* scalars, $\mathcal{L} = -\partial^\mu \phi_i^\dagger \partial_\mu \phi_i - m^2 \phi_i^\dagger \phi_i$, the same steps go through with unitary matrices in place of rotations, $U = e^{-i\theta^a T^a}$ with T^a hermitian (for unitary matrices the i is customarily put in the exponent, so that the generators are hermitian). The group is then $U(N) = U(1) \times SU(N)$, with $1 + (N^2 - 1)$ generators. These are the symmetry groups of the Standard Model.