Section 5: Feynman Diagrams in Position-Space

Wednesday, October 7th 2026

Srednicki Ch. 9–10; Schwartz Ch. 7; Dyson, PR **75**, 486 (1949); Nautilus, IAS, IAS 2, feynman.com

Contents

1 Into interaction territory **1**
- 1.1 Particles that finally meet . . . 1
- 1.2 Interaction correlation function . . . 2
- 1.3 The first example . . . 4
- 1.4 The bookkeeping problem . . . 6

2 Symmetry factors **6**
- 2.1 From pairings to diagrams . . . 6
- 2.2 Where the $1/S$ comes from . . . 6
- 2.3 The rules . . . 7
- 2.4 Two ways to get the number in front . . . 8

3 The diagram zoo **9**

4 Feynman rules in position-space **12**
- 4.1 The rules . . . 12
- 4.2 Example: the triangle, step by step . . . 12
- 4.3 Reading a diagram in time . . . 13
- 4.4 A bit of history . . . 14

1 Into interaction territory

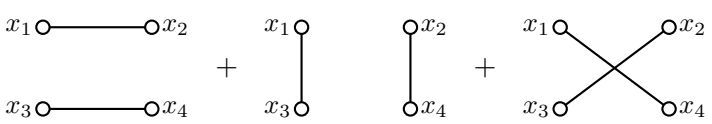
1.1 Particles that finally meet

Now that we are moving into interaction territory, things are going to become much more interesting! From the very first lecture, quantum field theory promised us particles that are created and annihilated. So far the free theory kept only half of that promise. The field $\phi(x)$ does create and destroy particles, but only at the points where *we* insert it; in between, the particles fly straight past each other without noticing. With interactions the theory itself starts creating and destroying particles.

Look at what the free theory gives for four points. From lecture (Wick's theorem),

$$\langle 0|\,\mathrm{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}\,|0\rangle_{\text{free}} = G_F(x_1,x_2)G_F(x_3,x_4) + G_F(x_1,x_3)G_F(x_2,x_4) + G_F(x_1,x_4)G_F(x_2,x_3), \tag{1}$$

with $G_F(x,y) = \langle 0|\,\mathrm{T}\{\phi(x)\phi(y)\}\,|0\rangle$ the Feynman propagator. In pictures:



Every diagram falls apart into two independent lines. One particle travels between two of the points, a second one between the other two, and they never meet. Nothing ever *happens*.

Interactions are terms in the Lagrangian with more than two fields. The simplest one is a cubic term, $\frac{1}{3!}g\phi^3$. Recall that ϕ is a sum of an annihilation part and a creation part, $\phi \sim a + a^\dagger$. So $\phi(z)^3$ contains terms like $a^\dagger a^\dagger a$: annihilate one particle at z and create two at the same point. It also contains $a^\dagger a\, a$ (two particles merge into one), $a^\dagger a^\dagger a^\dagger$ (three particles out of nothing), and so on. A cubic term in $\mathcal{L}$ is a *splitting point*, the place where the number of particles changes.

In lecture we wrote down the full theory,

$$\mathcal{L} = -\frac{1}{2} Z_\phi\, \partial^\mu\phi\, \partial_\mu\phi - \frac{1}{2} Z_m\, m^2\phi^2 + \frac{1}{3!} Z_g\, g\, \phi^3 + Y\phi, \tag{2}$$

with constants Z_ϕ, Z_m, Z_g and Y whose role we explain in a moment. Following Srednicki, we split it as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$:

$$\begin{aligned}
\mathcal{L}_0 &= -\frac{1}{2}\partial^\mu\phi\, \partial_\mu\phi - \frac{1}{2} m^2\phi^2, \\
\mathcal{L}_1 &= \frac{1}{3!} Z_g\, g\, \phi^3 + \mathcal{L}_{\text{ct}}, \\
\mathcal{L}_{\text{ct}} &= -\frac{1}{2}(Z_\phi - 1)\, \partial^\mu\phi\, \partial_\mu\phi - \frac{1}{2}(Z_m - 1)\, m^2\phi^2 + Y\phi.
\end{aligned} \tag{3}$$

Why split it this way? $\mathcal{L}_0$ is the free theory, the one we can solve exactly: its path integral is Gaussian. Everything else goes into $\mathcal{L}_1$ and is treated as a perturbation. $\mathcal{L}_1$ contains two types of terms.

- The *interaction*, $\frac{1}{3!}g\phi^3$: the genuinely new physics, with g a parameter we choose.
- The *counterterms*: the extra piece $\frac{1}{3!}(Z_g - 1)g\phi^3$ of the cubic term, and everything in $\mathcal{L}_{\text{ct}}$. They are written down only to cancel the infinities that appear in loop diagrams (we meet the first one in the example below), and their coefficients are not free: they are fixed, order by order in g, by conditions on the correlators. Y makes $\langle 0|\,\phi\,|0\rangle = 0$, Z_g fixes what we mean by g, and the other two fix the mass and the normalisation of the particle. Usually only a finite number of counterterms is needed, one for each term already in the Lagrangian (although not always: some theories need new ones at every order).

In principle we need all of them. For clarity, from now on we keep only the counterterm of g (inside $Z_g\, g$) and Y, and ignore Z_ϕ and Z_m (set them to 1). They come back in the diagram zoo. So for us

$$\mathcal{L}_1 = \frac{1}{3!} Z_g\, g\, \phi^3 + Y\phi, \qquad Z_g = 1 + \mathcal{O}(g^2), \quad Y = \mathcal{O}(g). \tag{4}$$

1.2 Interaction correlation function

A time-ordered correlator is the path integral with the fields inserted, divided by the same path integral without insertions,

$$\langle 0|\,\mathrm{T}\{\phi(x_1)\cdots\phi(x_n)\}\,|0\rangle = \frac{\int \mathcal{D}\phi\; \phi(x_1)\cdots\phi(x_n)\; e^{iS}}{\int \mathcal{D}\phi\; e^{iS}}, \qquad S = \int d^4x\, \mathcal{L}. \tag{5}$$

Split the action the same way as the Lagrangian, $S = S_0 + S_1$ (since $\phi(x)$ is an ordinary function, not an operator) with

$$S_0 = \int d^4x\, \mathcal{L}_0, \qquad S_1 = \int d^4z\, \mathcal{L}_1(z), \tag{6}$$

so that $e^{iS} = e^{iS_1}\, e^{iS_0}$.

Now divide the numerator and the denominator by the free path integral $\int \mathcal{D}\phi\, e^{iS_0}$:

$$\langle 0|\,\mathrm{T}\{\phi(x_1)\cdots\phi(x_n)\}\,|0\rangle = \frac{\dfrac{\int \mathcal{D}\phi\ \phi(x_1)\cdots\phi(x_n)\, e^{iS_1}\ e^{iS_0}}{\int \mathcal{D}\phi\ e^{iS_0}}}{\dfrac{\int \mathcal{D}\phi\ e^{iS_1}\ e^{iS_0}}{\int \mathcal{D}\phi\ e^{iS_0}}}. \tag{7}$$

Both the numerator and the denominator now have the form "something averaged with the free weight e^{iS_0}". Call this the *free correlator*,

$$\langle\cdots\rangle_0 \equiv \underbrace{\frac{\int \mathcal{D}\phi\ (\cdots)\ e^{iS_0}}{\int \mathcal{D}\phi\ e^{iS_0}}}_{\text{free path integral}}, \tag{8}$$

and the result is

$$\boxed{\langle 0|\,\mathrm{T}\{\phi(x_1)\cdots\phi(x_n)\}\,|0\rangle = \frac{\langle\phi(x_1)\cdots\phi(x_n)\ e^{iS_1}\rangle_0}{\langle e^{iS_1}\rangle_0}} \tag{9}$$

The gain of such a construction is that free correlators are Gaussian integrals, and we know everything about them from lecture:

- two fields give the propagator, $\langle\phi(x)\phi(y)\rangle_0 = G_F(x,y)$;
- any number of fields is given by

$$\langle\phi(y_1)\cdots\phi(y_N)\rangle_0 = \sum_{\text{pairings}}\ \prod_{\text{pairs }(a,b)} G_F(y_a,y_b), \tag{10}$$

  the sum running over all ways of splitting the N fields into pairs; for four fields it is the sum of three products we saw above, and for an odd number of fields it is zero;
- the free correlator is linear: $\langle c\int d^4z\, F(z)\rangle_0 = c\int d^4z\,\langle F(z)\rangle_0$;
- a power of the field at one point, like $\phi(z)^3 = \phi(z)\phi(z)\phi(z)$, counts as three separate fields that happen to sit at the same point.

What we cannot do is the path integral with e^{iS_1} in the exponent. So we expand it in a power series. Since S_1 is itself an integral, each power of S_1 comes with its own integration variable:

$$\begin{aligned} e^{iS_1} &= 1 + iS_1 + \frac{(iS_1)^2}{2!} + \frac{(iS_1)^3}{3!} + \dots \\ &= 1 + i\int d^4z\,\mathcal{L}_1(z) + \frac{i^2}{2!}\int d^4z\,d^4w\,\mathcal{L}_1(z)\mathcal{L}_1(w) + \frac{i^3}{3!}\int d^4z\,d^4w\,d^4u\,\mathcal{L}_1(z)\mathcal{L}_1(w)\mathcal{L}_1(u) + \dots \end{aligned} \tag{11}$$

Since $\mathcal{L}_1 = \frac{1}{3!}Z_g\, g\,\phi^3 + Y\phi$ is of order g, the V-th term of the series starts at order g^V. Inserting $\mathcal{L}_1$ explicitly, up to order g^2:

$$\begin{aligned} e^{iS_1} = 1 &+ \underbrace{i\int d^4z\left[\frac{1}{3!}Z_g\, g\,\phi(z)^3 + Y\phi(z)\right]}_{\text{order } g} \\ &+ \underbrace{\frac{i^2}{2!}\int d^4z\,d^4w\left[\frac{1}{3!}Z_g\, g\,\phi^3 + Y\phi\right]_z\left[\frac{1}{3!}Z_g\, g\,\phi^3 + Y\phi\right]_w}_{\text{order } g^2} + \mathcal{O}(g^3), \end{aligned} \tag{12}$$

where $[\cdots]_z$ means that every field inside sits at z. The recipe for any correlator is therefore: insert this expansion into Eq. (9), use linearity to pull out the constants and the integrals, and evaluate every free correlator.

1.3 The first example

Step 1: expand the numerator. Insert the expansion of e^{iS_1} into $\langle\phi(x_1)\phi(x_2)\phi(x_3)\,e^{iS_1}\rangle_0$, and abbreviate $\phi_i \equiv \phi(x_i)$:

$$\langle\phi_1\phi_2\phi_3\,e^{iS_1}\rangle_0 = \underbrace{\langle\phi_1\phi_2\phi_3\rangle_0}_{\text{order } g^0} + \underbrace{i\int d^4z\,\Big\langle\phi_1\phi_2\phi_3\Big[\frac{1}{3!}Z_g\,g\,\phi(z)^3 + Y\phi(z)\Big]\Big\rangle_0}_{\text{order } g^1}$$
$$+ \underbrace{\frac{i^2}{2!}\int d^4z\,d^4w\,\Big\langle\phi_1\phi_2\phi_3\,[\cdots]_z[\cdots]_w\Big\rangle_0}_{\text{order } g^2} + \mathcal{O}(g^3). \tag{13}$$

Count the fields in each free correlator. Order g^0: three fields. Order g^2: $3+3+3=9$, $3+3+1=7$ or $3+1+1=5$, depending on which terms of the two brackets we pick. All odd, so all zero. Only the order-g term survives:[1]

$$\langle\phi_1\phi_2\phi_3\,e^{iS_1}\rangle_0 = i\int d^4z\,\Big\langle\phi_1\phi_2\phi_3\Big[\frac{1}{3!}Z_g\,g\,\phi(z)^3 + Y\phi(z)\Big]\Big\rangle_0 + \mathcal{O}(g^3). \tag{14}$$

Step 2: the denominator. The same expansion with no external fields:

$$\langle e^{iS_1}\rangle_0 = 1 + \underbrace{i\int d^4z\,\Big\langle\frac{1}{3!}Z_g\,g\,\phi(z)^3 + Y\phi(z)\Big\rangle_0}_{\text{3 or 1 fields: 0}} + \mathcal{O}(g^2) = 1 + \mathcal{O}(g^2). \tag{15}$$

The order-g^2 part contains the vacuum bubbles we met in lecture. Here it does not matter: the numerator starts at order g, so dividing by $1+\mathcal{O}(g^2)$ only changes the answer at order g^3,

$$\frac{a\,g + \mathcal{O}(g^3)}{1+\mathcal{O}(g^2)} = a\,g + \mathcal{O}(g^3). \tag{16}$$

So the three-point function is the order-g term of the numerator:

$$\langle 0|\,\mathrm{T}\{\phi_1\phi_2\phi_3\}\,|0\rangle = \frac{iZ_g\,g}{3!}\int d^4z\,\langle\phi_1\phi_2\phi_3\,\phi(z)\phi(z)\phi(z)\rangle_0 + iY\int d^4z\,\langle\phi_1\phi_2\phi_3\,\phi(z)\rangle_0 + \mathcal{O}(g^3). \tag{17}$$

Step 3: count the pairings. The first free correlator contains six fields. How many ways are there to pair them up? Take the first field, $\phi(x_1)$: it can be paired with any of the other five, so 5 choices. Now take the first field that is still unpaired: it can be paired with any of the three that remain, so 3 choices. The last two fields have to be paired with each other, 1 choice. In total $5\cdot 3\cdot 1 = 15$ pairings. The same argument for N fields gives

$$(N-1)(N-3)\cdots 3\cdot 1 \equiv (N-1)!!\text{ pairings}. \tag{18}$$

The second correlator has four fields, so $3\cdot 1 = 3$ pairings.

Step 4: write out all the pairings. To keep track of which $\phi(z)$ is which, call the three copies ϕ_a, ϕ_b, ϕ_c for a moment (they all sit at z), and abbreviate $G_{ij} \equiv G_F(x_i,x_j)$, $G_{ia} \equiv G_F(x_i,z)$ for the copy a, and $G_{ab} \equiv G_F(z,z)$. Wick's theorem gives all 15 pairings:

$$\langle\phi_1\phi_2\phi_3\,\phi_a\phi_b\phi_c\rangle_0 = G_{1a}G_{2b}G_{3c} + G_{1a}G_{2c}G_{3b} + G_{1b}G_{2a}G_{3c}$$

[1] Its own $\mathcal{O}(g^2)$ part vanishes too, because Y contains only odd powers of g and Z_g only even ones. Both follow from a symmetry: $\mathcal{L}$ is unchanged under $\phi\to-\phi$ together with $g\to-g$, $Y\to-Y$.

$$
\begin{aligned}
&+ G_{1b}G_{2c}G_{3a} + G_{1c}G_{2a}G_{3b} + G_{1c}G_{2b}G_{3a} && \text{each } x_i \text{ to its own copy: } 6\\
&+ G_{12}(G_{3a}G_{bc} + G_{3b}G_{ac} + G_{3c}G_{ab}) && x_1 \text{ with } x_2\text{: } 3\\
&+ G_{13}(G_{2a}G_{bc} + G_{2b}G_{ac} + G_{2c}G_{ab}) && x_1 \text{ with } x_3\text{: } 3\\
&+ G_{23}(G_{1a}G_{bc} + G_{1b}G_{ac} + G_{1c}G_{ab}) && x_2 \text{ with } x_3\text{: } 3
\end{aligned} \tag{19}
$$

Indeed $6+3+3+3=15$. Now remember that a, b, c are all the same point z: every G_{ia}, G_{ib}, G_{ic} is $G_F(x_i,z)$ and every G_{ab}, G_{ac}, G_{bc} is $G_F(z,z)$. The six terms of the first two lines become equal, and so do the three terms in each bracket:

$$
\begin{aligned}
\langle\phi_1\phi_2\phi_3\,\phi(z)^3\rangle_0 = {}& 6\,G_F(x_1,z)G_F(x_2,z)G_F(x_3,z) + 3\,G_F(x_1,x_2)G_F(x_3,z)G_F(z,z)\\
&+ 3\,G_F(x_1,x_3)G_F(x_2,z)G_F(z,z) + 3\,G_F(x_2,x_3)G_F(x_1,z)G_F(z,z).
\end{aligned} \tag{20}
$$

The second correlator is simpler: two of the x's pair with each other and the third with $\phi(z)$,

$$
\langle\phi_1\phi_2\phi_3\,\phi(z)\rangle_0 = G_F(x_1,x_2)G_F(x_3,z) + G_F(x_1,x_3)G_F(x_2,z) + G_F(x_2,x_3)G_F(x_1,z). \tag{21}
$$

Step 5: assemble, and draw. Draw each G_F as a line, each external point x_i as an open circle, each $\phi(z)^3$ vertex as a dot, and each $Y\phi(z)$ vertex as a cross. Multiply by the prefactors and integrate over z:

$$
\begin{aligned}
&\langle 0|\,\mathrm{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\}\,|0\rangle\\
&\quad= \underbrace{\frac{iZ_g g}{3!}\cdot 6\int d^4z\, G_F(x_1,z)G_F(x_2,z)G_F(x_3,z)}\\
&\qquad+ \underbrace{\frac{iZ_g g}{3!}\cdot 3\,G_F(x_1,x_2)\int d^4z\, G_F(x_3,z)G_F(z,z)} + \underbrace{iY\,G_F(x_1,x_2)\int d^4z\, G_F(x_3,z)}\\
&\qquad+ \Big[\text{the same two terms with } x_3\leftrightarrow x_1 \text{ and with } x_3\leftrightarrow x_2\Big] \;+\; \mathcal{O}(g^3).
\end{aligned} \tag{22}
$$

These are the three diagrams from lecture. The first, the *vertex*, is a genuine interaction: three lines meet at one point z, integrated over all places where the meeting can happen.

Step 6: fix Y. In the lecture we discussed how the same steps for one point give the one-point function,

$$
\begin{aligned}
\langle 0|\,\phi(x)\,|0\rangle &= \frac{iZ_g g}{3!}\int d^4z\;\underbrace{\langle\phi(x)\,\phi(z)^3\rangle_0}_{3\,G_F(x,z)G_F(z,z)} + iY\int d^4z\;\underbrace{\langle\phi(x)\,\phi(z)\rangle_0}_{G_F(x,z)}\\
&= \int d^4z\,G_F(x,z)\Big[\frac{iZ_g g}{2}G_F(z,z) + iY\Big] + \mathcal{O}(g^3).
\end{aligned} \tag{23}
$$

Here $G_F(z,z)=G_F(0)$ does not depend on z (and is infinite). Demanding $\langle 0|\,\phi(x)\,|0\rangle = 0$ fixes

$$
Y = -\frac{Z_g\,g}{2}\,G_F(0) + \mathcal{O}(g^3) = -\frac{g}{2}\,G_F(0) + \mathcal{O}(g^3), \tag{24}
$$

the bracket vanishes, and the three-point function is a single diagram, with a tadpole and the counterterm canceling each other:

$$
\boxed{\langle 0|\,\mathrm{T}\{\phi(x_1)\phi(x_2)\phi(x_3)\}\,|0\rangle = iZ_g\,g\int d^4z\,G_F(x_1,z)G_F(x_2,z)G_F(x_3,z) + \mathcal{O}(g^3)} \tag{25}
$$

1.4 The bookkeeping problem

In the first example we could sort all $15 + 3 = 18$ pairings by hand into three kinds of picture. That luxury does not last. For the two-point function at order g^V (ignoring Y) there are $N = 2 + 3V$ fields, and $(N-1)!!$ pairings:

order	g^2	g^4	g^6
pairings	$7!! = 105$	$13!! = 135\,135$	$19!! = 654\,729\,075$

Nobody sums 654 million pairings. But, as in the example, huge groups of pairings give the same integral, and only a handful of different *diagrams* appear.

2 Symmetry factors

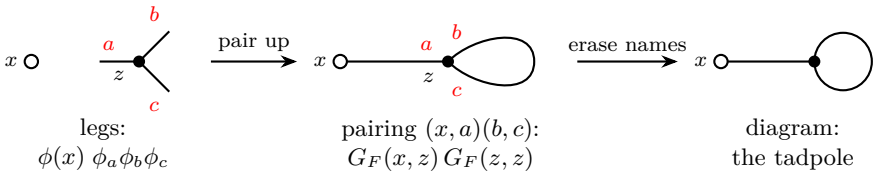
To keep the counting clean we ignore the counterterm $Y\phi$ in this section, so the only vertex is the cubic one, from $\frac{1}{3!} Z_g\, g\, \phi^3$.

2.1 From pairings to diagrams

The term with V vertices in the numerator of the one formula is

$$\underbrace{\frac{1}{V!}\Big(\frac{iZ_g g}{3!}\Big)^V}_{\text{prefactor}} \int d^4z_1 \cdots d^4z_V \, \Big\langle \phi(x_1)\cdots\phi(x_n) \, \underbrace{\phi(z_1)\phi(z_1)\phi(z_1)}_{\text{vertex } z_1} \cdots \underbrace{\phi(z_V)\phi(z_V)\phi(z_V)}_{\text{vertex } z_V} \Big\rangle_0 . \tag{26}$$

As in the first example, name the three fields at each vertex a, b, c: the three *legs* of the vertex. A pairing joins the legs two by two, each pair is a line G_F, and the *diagram* is what is left after erasing the names (the x_i keep theirs):

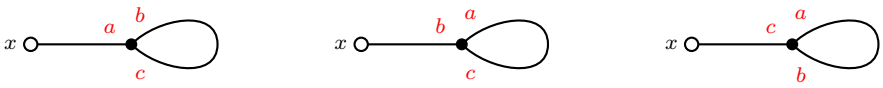


The value of a pairing depends only on its diagram. The question is how many pairings give the same diagram.

2.2 Where the $1/S$ comes from

Renaming the vertices ($V!$ ways) or the legs at each vertex (3! ways each) does not change the value: the z's are integration variables, and the legs of a vertex sit at the same point. If all $V!\,(3!)^V$ renamings gave different pairings, they would cancel the prefactor exactly. They do not always.

The tadpole. The $3! = 6$ renamings of a, b, c give only 3 pairings:



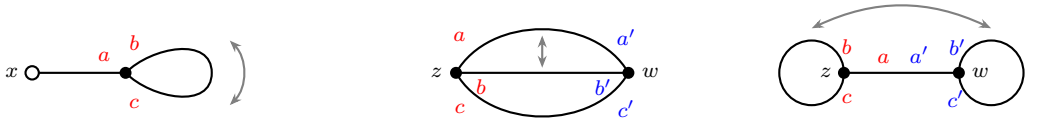
$(x,a)(b,c)$ $(x,b)(a,c)$ $(x,c)(a,b)$

since $b \leftrightarrow c$ gives back the same pairing: (b,c) and (c,b) are the same line. The renamings that give back the same pairing (doing nothing included) are the *symmetries*; their number is the *symmetry factor* S. Here $S=2$, and in general

$$\#\text{ pairings} = \frac{V!\,(3!)^V}{S} \quad\Longrightarrow\quad \boxed{D = \frac{(iZ_g g)^V}{S}\int \prod_{\text{vertices}} d^4z \prod_{\text{lines}} G_F} \tag{27}$$

(S is not the action). For the tadpole, $\frac{iZ_g g}{3!}\cdot 3 = \frac{iZ_g g}{2}$.

Which renamings are symmetries. Note that the external points are never renamed! A renaming gives back the same pairing only by



loops: $b \leftrightarrow c$ **multiple lines**: $a \leftrightarrow b$, $a' \leftrightarrow b'$ **exchanges**: $z \leftrightarrow w$, $\alpha \leftrightarrow \alpha'$

$$\begin{aligned} \text{loops:}\quad & (x,a)(b,c) \to (x,a)(c,b),\\ \text{multiple lines:}\quad & (a,a')(b,b')(c,c') \to (b,b')(a,a')(c,c'),\\ \text{exchanges:}\quad & (a,a')(b,c)(b',c') \to (a',a)(b',c')(b,c), \end{aligned}$$

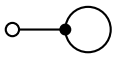
Anything else, e.g. $a \leftrightarrow b$ at z alone in the middle diagram, turns $(a,a')(b,b')$ into $(b,a')(a,b')$: a different pairing.

2.3 The rules

The symmetries come in only three types, and S can be read off the diagram:

Start. Hold every external point x_i fixed: external points never move. Begin with $S=1$.

Rule 1 (loops). For every line that starts and ends on the same vertex: $S \to 2S$.
(Its two legs can be swapped.)



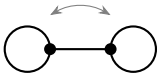
×2

Rule 2 (multiple lines). For every set of k lines joining the same two vertices: $S \to k!\,S$.
(The lines can be permuted.)



×2!

Rule 3 (exchanges). Count the permutations of the vertices, each carrying everything attached to it, that give back the same diagram (including leaving every vertex in place); multiply S by that number.
(A vertex joined to an external point can never be exchanged.)



×2
with rule 1:
$S = 2\cdot 2\cdot 2 = 8$

S is the product of all the factors.

Shortcut. A tree diagram (no closed loops) whose ends are all external points has $S = 1$: any move would have to swap two external points.

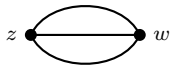
2.4 Two ways to get the number in front

There are now two independent ways to find the number in front of a diagram D:

(a) *Brute force.* Count the pairings that give D, and multiply by the prefactor $\frac{1}{V!}\left(\frac{iZ_g g}{3!}\right)^V$. This always works, but it is slow.

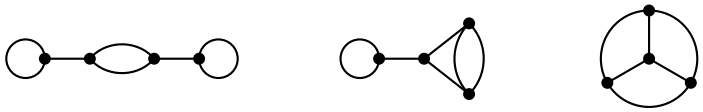
(b) *Shortcut.* Read S off the diagram with the three rules; the number is $\frac{(iZ_g g)^V}{S}$. Fast, but easy to get wrong.

Example: the theta. $V = 2$, no external points: a vacuum bubble, two vertices joined by three lines.



(a) Brute force. In $\langle\phi(z)^3\phi(w)^3\rangle_0$ (15 pairings in all), the three legs a, b, c of z must each pair with a leg of w: $3 \cdot 2 \cdot 1 = 6$ pairings, so $\frac{1}{2!}\left(\frac{iZ_g g}{3!}\right)^2 \cdot 6 = \frac{(iZ_g g)^2}{12}$.
(b) Shortcut. No self-closing loops; three lines between z and w (rule 2, factor 3!); exchanging z and w gives back the same diagram (rule 3, factor 2). $S = 12$.

Exercise for the board. Here are three of the connected vacuum bubbles with four cubic vertices.

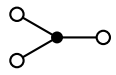


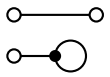
(A) (B) (C) Mercedes

Find their symmetry factors. (Answer: $S_A = 16$: two loops, a double line and the reflection, $2 \cdot 2 \cdot 2! \cdot 2$. $S_B = 8$: one loop, a double line, and the exchange of the two vertices it joins, $2 \cdot 2! \cdot 2$. $S_C = 24$: every permutation of the four vertices is a symmetry, 4!.)

3 The diagram zoo

Connected and disconnected. A diagram is *connected* if one can walk from any point of it to any other along lines. Otherwise it is *disconnected*: a product of several connected pieces.


connected


disconnected

Disconnected diagrams are products. Take a diagram with V vertices made of two pieces, with V_1 and $V_2 = V - V_1$ vertices. Which of the V vertex names sit in the first piece does not matter (they are dummy variables), and there are $\binom{V}{V_1} = \frac{V!}{V_1!\,V_2!}$ ways to choose them. So

$$\frac{1}{V!} \cdot \frac{V!}{V_1!\,V_2!} = \frac{1}{V_1!} \cdot \frac{1}{V_2!}\,: \tag{28}$$

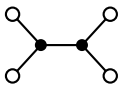
each piece comes with its own prefactor, and a disconnected diagram is simply the product of its connected pieces. A disconnected diagram describes groups of particles that never meet; it carries no new information. The connected diagrams are the interesting objects.

Trees

Trees and loops. Count the *internal lines* I (lines with no end on an external point) and the *vertices* V (dots and crosses alike). A connected diagram, with at least one vertex, has

$$L = I - V + 1$$

independent closed loops. A *tree* has $L = 0$, a one-loop diagram $L = 1$, and so on.


tree, $L = 0$

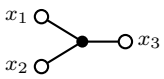

one loop, $L = 1$

Why $L = I - V + 1$. External lines never lie on a loop, so ignore them. Start from the V vertices alone, V separate pieces, and add the I internal lines one at a time. Each new line either joins two separate pieces into one, or connects two vertices that are already connected and closes a new loop. To end up with one piece, exactly $V - 1$ lines do the joining; the other $I - (V - 1)$ each close a loop. For the bubble, $V = I = 2$:



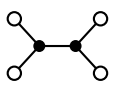
2 pieces joined: 1 piece one loop closed

so $L = 2 - 2 + 1 = 1$.



The vertex. $\mathcal{O}(g)$, $L = 0$, $S = 1$.

$$ig \int d^4z\; G_F(x_1, z) G_F(x_2, z) G_F(x_3, z)$$

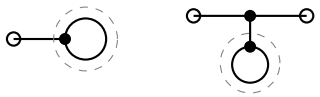


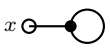
Exchange tree. $\mathcal{O}(g^2)$, $L = 0$, $S = 1$. Two particles meet at z, and a third one is exchanged between z and w. Three channels, according to which point shares a vertex with x_1:

$$(ig)^2 \int d^4z\, d^4w\; G_F(x_1, z) G_F(x_2, z) G_F(z, w) G_F(w, x_3) G_F(w, x_4) + (x_2 \leftrightarrow x_3) + (x_2 \leftrightarrow x_4)$$

The tadpole and its counterterm

What a tadpole is. A *tadpole* is a piece with no external points that hangs from the rest of the diagram by a single line. The simplest one is the one-point diagram below.





Tadpole. $\mathcal{O}(g)$, $L=1$, $S=2$. The loop $G_F(z,z)$ is a particle created and absorbed at the same point, and it is infinite.[2]

$$\frac{ig}{2}\int d^4z\, G_F(x,z)G_F(z,z)$$

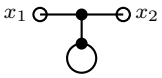


Its counterterm: the Y vertex. $\mathcal{O}(g)$, $L=0$, $S=1$. With the tadpole it forms the contribution $\frac{ig}{2}G_F(0)+iY$ to the 1-point correlator, which vanishes once Y is tuned.

$$iY\int d^4z\, G_F(x,z)$$

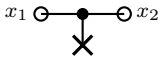
Tadpoles are cancelled by Y. The same cancellation works for every tadpole. Y is tuned order by order so that $\langle\phi\rangle=0$. Summing over everything that can hang from a single line attached to the rest of a diagram (all tadpoles and all crosses) gives the one-point function with its last line $G_F(x,z)$ stripped off, and $\langle\phi\rangle=0$ makes it vanish. So diagrams containing tadpoles and crosses can be dropped altogether, once Y is tuned.

Propagator corrections and their counterterms



Tadpole on a propagator[3]**.** $\mathcal{O}(g^2)$, $L=1$, $S=2$.

$$\tfrac{(ig)^2}{2}\iint G_F(x_1,z)G_F(z,x_2)G_F(z,w)G_F(w,w)$$

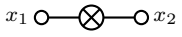


Its counterterm: a cross on a propagator. $\mathcal{O}(g^2)$, $L=0$, $S=1$.

$$ig\,iY\iint G_F(x_1,z)G_F(z,x_2)G_F(z,w)$$



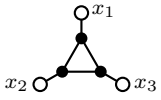
Bubble (self-energy). $\mathcal{O}(g^2)$, $L=1$, $S=2$. The particle splits into two and recombines: the first correction to free propagation.



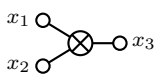
Its counterterm: the two-leg vertex. $\mathcal{O}(g^2)$, $L=0$, $S=1$. Here the counterterms we have been ignoring come back. The bubble is infinite, and cancelling it (and fixing the normalisation) needs the terms $-\frac{1}{2}(Z_\phi-1)\partial^\mu\phi\,\partial_\mu\phi-\frac{1}{2}(Z_m-1)m^2\phi^2$ of $\mathcal{L}_{\rm ct}$, with $Z_\phi-1$, Z_m-1 of order g^2. They give a vertex with two legs (we mark every Z-counterterm insertion with $\otimes$).

$$-i\int d^4z\left[(Z_\phi-1)\,\frac{\partial G_F(x_1,z)}{\partial z^\mu}\frac{\partial G_F(z,x_2)}{\partial z_\mu}+(Z_m-1)\,m^2\,G_F(x_1,z)G_F(z,x_2)\right]$$

The vertex correction and its counterterm



Triangle. $\mathcal{O}(g^3)$, $L=1$, $S=1$. The first loop correction to the vertex.



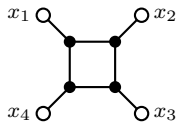
Its counterterm: the Z_g-1 part of the cubic vertex. $\mathcal{O}(g^3)$, $L=0$, $S=1$. Write $Z_g\,g=g+(Z_g-1)\,g$; the second piece is a vertex diagram of the same order as the triangle. Z_g is fixed by the condition that defines what we mean by g.

$$i(Z_g-1)\,g\int d^4z\, G_F(x_1,z)G_F(x_2,z)G_F(x_3,z)$$

[2]The name comes from Coleman and Glashow's 1964 *Physical Review* paper. Coleman later claimed, perhaps as a joke, that the editors had rejected his first choice, "spermion".

[3]I vaguely remember people calling this diagram a balloon diagram, but I cannot find any trustworthy evidence to it.

Four points at one loop

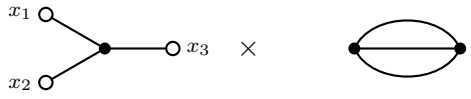


Box. $\mathcal{O}(g^4)$, $L = 1$, $S = 1$. The box has no counterterm: $\mathcal{L}$ has no ϕ^4 term, and none is needed, since the box is finite.

$$(ig)^4 \int d^4a\, d^4b\, d^4c\, d^4d\; G_F(x_1,a)G_F(x_2,b)G_F(x_3,c)G_F(x_4,d)$$
$$\times\, G_F(a,b)G_F(b,c)G_F(c,d)G_F(d,a) \;+\; (2 \text{ other orders})$$

Vacuum bubbles

Last come the *vacuum bubbles*: pieces with no external points and no line to the rest of the diagram. Nothing goes in, nothing comes out. They make up the denominator, and they cancel with the numerator. In lecture we saw this at order g^2; the argument works at every order. Split the diagram of any pairing in the numerator into the *external part*, the vertices connected (through some chain of lines) to at least one external point, and the *vacuum part*, the rest:



external part, $V_1 = 1$ vacuum part, $V_2 = 2$

Each part comes with its own prefactor, as for any disconnected diagram, so the sum over all V and all pairings factorises:

$$\begin{aligned}\langle\phi(x_1)\cdots\phi(x_n)\,e^{iS_1}\rangle_0 &= \sum_{V_1,V_2}\Big[\frac{1}{V_1!}\int\langle\phi(x_1)\cdots\phi(x_n)\,(i\mathcal{L}_1)^{V_1}\rangle_0\Big|_{\text{no vacuum part}}\Big]\Big[\frac{1}{V_2!}\int\langle(i\mathcal{L}_1)^{V_2}\rangle_0\Big]\\ &= \Big[\sum_{V_1}\frac{1}{V_1!}\int\langle\phi(x_1)\cdots\phi(x_n)\,(i\mathcal{L}_1)^{V_1}\rangle_0\Big|_{\text{no vacuum part}}\Big]\times\langle e^{iS_1}\rangle_0. \qquad (29)\end{aligned}$$

The second factor is exactly the denominator, so

$$\boxed{\langle 0|\,\mathrm{T}\{\phi(x_1)\cdots\phi(x_n)\}\,|0\rangle = \text{sum of all diagrams with no vacuum part}} \qquad (30)$$

The order-g^2 bubbles. At order g^2 the cubic vertices give two vacuum bubbles, the dumbbell and the theta. Each comes with counterterm partners, in which a tadpole or a self-energy bubble inside it is replaced by the corresponding counterterm vertex.



Dumbbell. $\mathcal{O}(g^2)$, $L = 2$, $S = 8$.



Theta. $\mathcal{O}(g^2)$, $L = 2$, $S = 12$.



Counterterm: one tadpole replaced by a cross. $\mathcal{O}(g^2)$, $L = 1$, $S = 2$.



Counterterm: the bubble inside the theta replaced by $\otimes$. $\mathcal{O}(g^2)$, $L = 1$, $S = 2$. The theta is a self-energy bubble closed by a third line; replacing the bubble by the two-leg vertex leaves a line from that vertex to itself.

Counterterm: both tadpoles replaced by crosses. $\mathcal{O}(g^2)$, $L = 0$, $S = 2$.

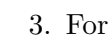
In practice we keep only *connected diagrams without tadpoles, crosses or vacuum bubbles*. The dropped ones still deserve the look we gave them: they fill every intermediate step, and their cancellations against the counterterms are where renormalization starts. Everything is now in place to state the Feynman rules.

4 Feynman rules in position-space

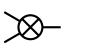
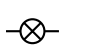
4.1 The rules

Collecting the vertices, the symmetry factors and the choice of which diagrams matter, the connected correlator $\langle 0|\,\mathrm{T}\{\phi(x_1)\cdots\phi(x_n)\}\,|0\rangle_{\mathrm{conn}}$ is computed in two steps: first the diagrams built from the interaction $\frac{1}{3!}g\phi^3$ alone, then the counterterm diagrams that come with them.

Main rules.

1. Draw the external points $x_1,\dots,x_n$ and V cubic vertices (dots), and join them by lines in all topologically distinct ways such that each x_i has one line, each dot has three, and the diagram is connected. This gives the diagrams of order g^V.
2. For each line between a and b, write $G_F(a,b)$.
3. For each dot, write  $= ig\int d^4z$.
4. Divide by the symmetry factor S.
5. Add up all the diagrams, leaving out those with tadpoles.

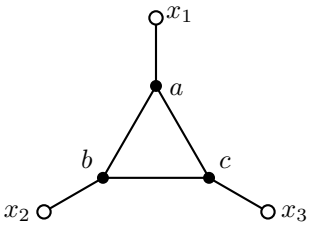
Counterterm rules. Each counterterm is one more kind of vertex, used exactly like the dot: it is joined to the rest of the diagram by as many lines as it has legs, it is counted like any other vertex in the symmetry factor (identical counterterm vertices can be exchanged by rule 3), and it brings its own power of g. At a given order, add every diagram in which some vertices or loops are replaced by counterterm vertices of the same total order.

vertex	factor	order	fixed by, and what it cancels
	$iY\int d^4z$	g	$\langle 0\|\,\phi\,\|0\rangle = 0$; cancels the tadpoles
	$i(Z_g-1)\,g\int d^4z$	g^3	the definition of g; cancels the infinity of the vertex correction
	$-i\int d^4z\,[(Z_\phi-1)\,\partial_\mu\cdots\partial^\mu + (Z_m-1)\,m^2]$	g^2	the normalization of ϕ and the definition of m; cancels the infinity of the self-energy

In the two-leg vertex the two derivatives act on the two attached propagators. The cubic counterterm is the Z_g-1 part of $\frac{1}{3!}Z_g\,g\,\phi^3$: with $Z_g = 1+\mathcal{O}(g^2)$, the main rules use the leading piece ig, and the rest becomes a vertex of its own. Once Y is tuned, the crosses and the tadpoles cancel each other, so in practice both are left out (rule 5), and only the two Z counterterms remain.

4.2 Example: the triangle, step by step

To see the rules at work, take the triangle from the zoo and go through them one by one.



1. *Draw and name.* External points x_1, x_2, x_3; three cubic vertices, named a, b, c after the external point they touch. Lines: x_1a, x_2b, x_3c, ab, bc, ca. Order g^3.
2. *Lines.* $G_F(x_1, a)\, G_F(x_2, b)\, G_F(x_3, c)\, G_F(a, b)\, G_F(b, c)\, G_F(c, a)$.
3. *Vertices.* $(ig)^3 \int d^4a\, d^4b\, d^4c$.
4. *Symmetry factor.* Rule 1: no line starts and ends on the same vertex. Rule 2: no two lines join the same pair of vertices. Rule 3: every vertex touches an external point, so none can be exchanged. $S = 1$.
5. *Result.*
$$(ig)^3 \int d^4a\, d^4b\, d^4c\; G_F(x_1, a) G_F(x_2, b) G_F(x_3, c)\, G_F(a, b) G_F(b, c) G_F(c, a). \tag{31}$$
Any other assignment of x_1, x_2, x_3 to the corners of the triangle gives the same diagram, so there is only this one diagram.

4.3 Reading a diagram in time

So far a diagram has been bookkeeping for pairings. It turns out that it also describes particles moving through spacetime! Go back to the simplest interaction diagram, three lines meeting at z,

$$ig \int d^4z\; G_F(x_1, z) G_F(x_2, z) G_F(x_3, z). \tag{32}$$

Recall how to read G_F: the time-ordered product makes the earlier field act first, so $G_F(a, b)$ is the amplitude for a particle to be created at the earlier of the two points and absorbed at the later one. The integral over z runs over all spacetime points, so it contains every possible time order. Pick the times of the external points as $x_3^0 < x_1^0, x_2^0$. When $x_3^0 < z^0 < x_1^0, x_2^0$, the diagram describes one particle created at x_3 that splits at z into two, which are absorbed at x_1 and x_2. When z^0 is earlier than all three, three particles are created at z out of nothing; one of them is absorbed at x_3, and the other two at x_1 and x_2. To draw this we follow the usual convention for Feynman diagrams: time runs from left to right, so whatever happens earlier sits further to the left (incoming particles on the left, outgoing ones on the right). The vertical direction means nothing; it only separates the lines.



$x_3^0 < z^0 < x_1^0, x_2^0$:
one particle splits into two

$z^0 < x_3^0 < x_1^0, x_2^0$:
three particles out of the vacuum

One Feynman diagram contains all these processes at once, with the right relative amplitudes: the time order is summed over, not chosen. The arrows are only a reading aid; the lines themselves carry no direction.

4.4 A bit of history

The rules we derived in a few pages were far from obvious at the time. In the spring of 1948 Feynman presented his diagrams at a small conference in the Pocono Mountains, and almost nobody could follow him: he had no derivation, only diagrams and rules that seemed to come out of nowhere. At the same meeting Julian Schwinger presented his own approach: a manifestly covariant operator formalism (field operators and a Lorentz-invariant version of the perturbation expansion), rigorous but extremely heavy. Independently, Sin-Itiro Tomonaga had developed an equivalent covariant formulation in Japan.

The person who made the diagrams usable by everyone was Freeman Dyson, then a 24-year-old graduate student at Cornell. That summer he rode with Feynman from Cleveland to Albuquerque, and then spent six weeks at the summer school in Ann Arbor, Michigan, listening to Schwinger lecture on his formalism. After a short visit to Berkeley he took the Greyhound bus back east, and somewhere in Nebraska he realised that the two were the same theory, and that Feynman's rules could be *derived*, with all the factors fixed, from the Schwinger–Tomonaga machinery. His 1949 paper, "The radiation theories of Tomonaga, Schwinger, and Feynman", did exactly that, and the diagrams then spread through the physics world with the young postdocs who learned them from Dyson at the Institute for Advanced Study in Princeton. Dyson never earned a PhD: on the strength of this work Cornell made him a full professor in 1951. Feynman, Schwinger and Tomonaga shared the 1965 Nobel Prize for quantum electrodynamics. Feynman remained fond of the diagrams for life: in 1975 he bought a van and had them painted all over it.



Figure 1: Feynman's 1975 Dodge Tradesman Maxivan, painted with Feynman diagrams (licence plate QANTUM). After his death Gweneth Feynman sold it to Ralph Leighton; it was later restored by Seamus Blackley and exhibited at Fermilab in 2014. See feynman.com