# Section 6: LSZ, Feynman Rules in Momentum-Space, Cross Section & Decay Rate

Wednesday, October 14th 2026

Srednicki Ch. 5, 10, 11; Schwartz Ch. 5–7; 2023 section notes by X. Y. Zhang; PDG kinematics review

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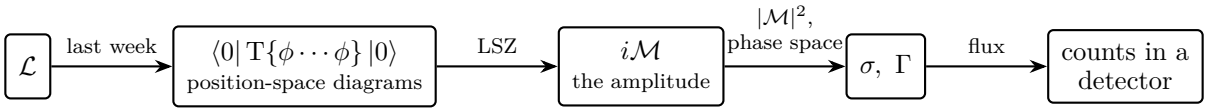
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# 1 Intro

Last week we learned to turn a Lagrangian into time-ordered correlators: draw the diagrams, write a propagator $G_F(x,y)$ for every line and $ig\int d^4z$ for every vertex, divide by the symmetry factor, add everything up. The trouble is that nobody has ever measured a correlator. The quantities that come out of an experiment are of two kinds: *cross sections* $\sigma$ (how often two particles that are thrown at each other do something) and *decay rates* $\Gamma$ (how fast an unstable particle disappears). Today we go from the correlator to these two numbers:




# 2 The LSZ formula

## 2.1 The formula and its parts

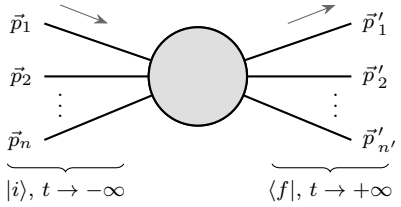
A scattering experiment prepares $n$ particles with momenta $\vec{p}_1,\dots,\vec{p}_n$ far apart in the distant past, lets them meet, and finds $n'$ particles with momenta $\vec{p}\,'_1,\dots,\vec{p}\,'_{n'}$ in the distant future. The amplitude for that is

$$\underbrace{\langle \text{out}:\vec{p}\,'_1\cdots\vec{p}\,'_{n'}\,|\,\text{in}:\vec{p}_1\cdots\vec{p}_n\rangle}_{\text{(a)}} = \underbrace{i^{\,n+n'}\int d^4x_1\,e^{ip_1x_1}\cdots\int d^4x_n\,e^{ip_nx_n}\int d^4x'_1\,e^{-ip'_1x'_1}\cdots\int d^4x'_{n'}\,e^{-ip'_{n'}x'_{n'}}}_{\text{(b)}}$$
$$\times\underbrace{\left(-\partial_1^2+m_1^2\right)\cdots\left(-\partial_n^2+m_n^2\right)\left(-\partial_1'^2+m_1'^2\right)\cdots\left(-\partial_{n'}'^2+m_{n'}'^2\right)}_{\text{(c)}}\ \underbrace{\langle 0|\,\mathrm{T}\{O'_1(x'_1)\cdots O'_{n'}(x'_{n'})\,O_1^\dagger(x_1)\cdots O_n^\dagger(x_n)\}\,|0\rangle}_{\text{(d)}} \tag{1}$$

Here $px = p_\mu x^\mu = -Et+\vec{p}\cdot\vec{x}$, every momentum is *on shell*, $p_i^2=-m_i^2$ as for a free particle, and there is one plane wave and one Klein–Gordon operator per particle, in or out. From now on we abbreviate the left-hand side as $\langle f|i\rangle$ ($f$ for the final state, $i$ for the initial one).

**(a) What is computed.**

**⇒ process with incoming/outgoing particles with momenta**




$\langle f|i\rangle$ is the amplitude that the particles prepared in the far past, $|i\rangle$, are found in the far future as the out-particles $\langle f|$; both sets are far apart and on shell, so they are just free particles. $|\langle f|i\rangle|^2$ is, up to the normalization of the states, the probability of the process.

Equivalently, write both as free-particle states and let $S$ be the operator that evolves the system from $t = -\infty$ to $t = +\infty$: then $\langle f|i\rangle = \langle f|S|i\rangle$, a matrix element of $S$, or an *S-matrix element*. If the particles never interact, $S = 1$ and nothing happens; everything else, the piece that carries the physics, is $\mathcal{M}$, Eq. (4).

**(b) Plane waves.**

**⇒ want momenta, not position**[1]

$$x \longleftrightarrow p$$

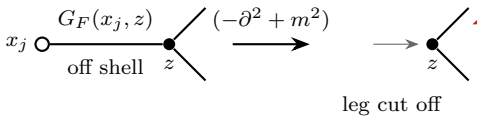
$$\underset{\text{positions}}{\langle 0|\,\mathrm{T}\{O(x_1)\cdots\}\,|0\rangle} \int d^4x\, e^{ipx} \longleftrightarrow \text{momenta: the experiment}$$

The correlator knows positions; the experiment fixes momenta. $e^{ipx} = e^{-iEt+i\vec{p}\cdot\vec{x}}$ is the wavefunction of a particle of momentum $\vec{p}$, so integrating $x_j$ against it projects the particle created at $x_j$ onto momentum $\vec{p}_j$; an outgoing particle gets the complex conjugate, $e^{-ip'x'}$. In a diagram that sign is the difference between momentum flowing in and flowing out.

*progression of actions on the correlator*

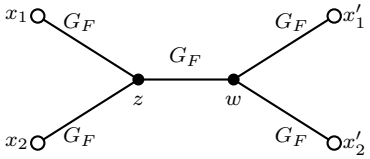
**(c) Klein–Gordon operators.**

**⇒ want incoming/outgoing real particles**




In a diagram each external point $x_j$ is joined to the rest by a propagator $G_F(x_j, z)$, a particle travelling from $x_j$ to the first vertex is *off shell*, with every $p^2$ allowed. The experiment sends in a *real* particle, $p^2 = -m^2$, and we want only that part. Since $(-\partial^2 + m^2)G_F = -i\delta^4$, the operator collapses the external propagator; in momentum space it cancels the pole $1/(p^2 + m^2)$ and keeps its coefficient, the residue. Cutting the external propagators off a diagram is called *amputating* it.

**(d) The correlator.**




one position-space diagram: $G_F$ on every line, $ig\int d^4z$ at every dot; the correlator is the sum of all

The same object as last week: the sum of all position-space diagrams with $n + n'$ external points. $O_j$ is any operator that destroys particle $j$ with unit strength, $\langle 0|O_j(x)|\vec{p}_j\rangle = e^{ip_j x}$, and $O_j^\dagger$ creates it, hence the daggers on the incoming legs; for us $O_j$ is the field of that particle, one per particle.

**In lecture notation.** Eq. (1) is lecture Eq. (224). There the vacuum of the interacting theory is $|\Omega\rangle$ (our $|0\rangle$), there are $n$ incoming and $m$ outgoing particles (our $n'$), and each may carry extra labels $\sigma$ (spin, species). $O^\dagger$ is called an *interpolating operator*: any operator with $\langle\vec{p}|O^\dagger(x)|\Omega\rangle = Z^* e^{-ipx} \neq 0$ and $\langle\Omega|O^\dagger|\Omega\rangle = 0$, Eqs. (204)–(208); our $O_j$ is the lecture's $O_j/Z_j$, which is why Eq. (1) has no $1/Z$ factors (for $O = \phi$ at tree level $Z = 1$).

[1] Why momenta? (i) The experiment prepares and detects particles of definite momentum, never a field at a point, so the question is posed in momentum space. (ii) The integrals are easy there: a propagator becomes the rational function $-i/(p^2 + m^2 - i\epsilon)$ and each vertex integral $\int d^4z\, e^{i(\sum p)z}$ becomes a delta function imposing momentum conservation, so a tree diagram has no integral left at all. (iii) Momentum is conserved, so a state of definite $\vec{p}$ reaches a far-away detector unchanged.

## 2.2 Why on shell

Why does the residue at the pole, and not anything else, describe the particles in the detector? Think of what the field does to the vacuum once there are interactions:

$$\phi|0\rangle = \underbrace{1 \text{ particle}}_{\substack{E_{\vec{p}}=\sqrt{|\vec{p}|^2+m^2} \\ \text{for a given } \vec{p}}} + \underbrace{2 \text{ particles}}_{E \geq \sqrt{|\vec{p}|^2+(2m)^2}} + 3 \text{ particles} + \cdots \tag{2}$$

For a given total $\vec{p}$ the one-particle state has a single energy, while two or more particles can share the momentum in many ways and so cover a continuous range of energies. In the Fourier transform of the two-point function each piece shows up with its own energy:

$$\tilde{G}_F(p) = \underbrace{\frac{-i}{p^2+m^2-i\epsilon}}_{\substack{1 \text{ particle:} \\ \text{pole at } p^2=-m^2}} + \underbrace{\int_{4m^2}^{\infty} d\mu^2\, \rho(\mu^2)\, \frac{-i}{p^2+\mu^2-i\epsilon}}_{\substack{2,\,3,\,\ldots \text{ particles:} \\ \text{continuum for } -p^2 \geq 4m^2}}, \tag{3}$$

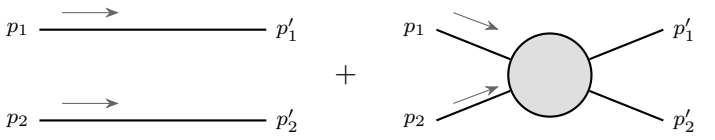
where $\rho(\mu^2) \geq 0$ says how much of each total mass $\mu$ the field creates. Now let time run. The one-particle piece has a single energy and stays what it is. The continuum is a superposition $\int dE\, c(E)\, e^{-iEt}|E\rangle$ of many energies, and as $t \to \pm\infty$ its components dephase and cancel against each other: physically, it is a cloud of particles flying apart, and in the far past or future they are nowhere near the collision. A detector far away sees only the one-particle piece; the in- and out-states are built from it, and LSZ keeps the residue at the pole and throws the continuum away.[2]

## 2.3 The amplitude

The S-matrix element always splits into two parts,

$$\langle f|i\rangle = \underbrace{\langle f|i\rangle_{\text{free}}}_{\text{nothing happens}} + i\,(2\pi)^4\delta^4\Big(\textstyle\sum p_{\text{in}} - \sum p_{\text{out}}\Big)\,\mathcal{M}. \tag{4}$$

The first term comes from the disconnected diagrams, in which every particle flies straight through, the second from the connected ones:




disconnected: $\langle f|i\rangle_{\text{free}}$ connected: $i(2\pi)^4\delta^4(\cdots)\,\mathcal{M}$

Where does the delta function come from? From translation invariance. The correlator depends only on differences of positions, so shifting every $x$ and $x'$ by the same $a$ leaves it unchanged, while the plane waves in (b) pick up a factor $e^{i(\sum p_{\text{in}} - \sum p_{\text{out}})a}$; the integral over that common shift is then $\int d^4a\, e^{i(\sum p_{\text{in}} - \sum p_{\text{out}})a} = (2\pi)^4\delta^4(\sum p_{\text{in}} - \sum p_{\text{out}})$. Everything else is $i\mathcal{M}$:

> $i\mathcal{M}$ is the sum of the *connected* diagrams, amputated (external legs cut off, their momenta put on shell), with the overall $(2\pi)^4\delta^4$ removed.

[2] Lecture §6.2.3 makes the same argument with wave packets: the creation operator acting on the vacuum gives one particle plus multiparticle states, Eq. (213), and the multiparticle overlap carries a phase $e^{i(q^0-k^0)t}$ with $q^0 - k^0 < 0$, Eq. (220), which oscillates and averages to zero as $t \to \pm\infty$.

# 3 Main example

## 3.1 The theory: $\phi^3$

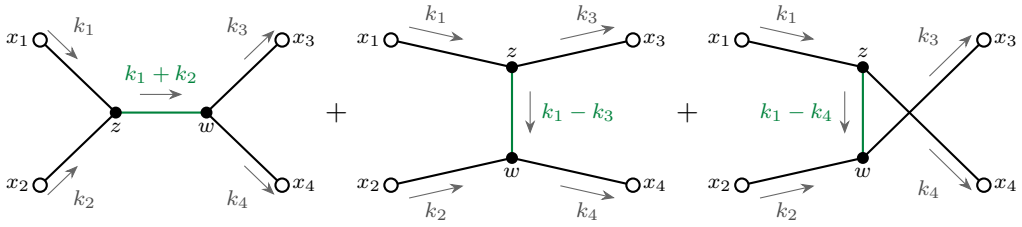
Take one real scalar $\phi$ of mass $m$, with the free Lagrangian as always and the cubic interaction

$$\mathcal{L}_1 = \frac{g}{3!}\,\phi^3. \tag{5}$$

The vertex has three legs, and the vertex factor is $ig$: the 3! is cancelled by the 3! ways of attaching three identical legs. All lines are the same particle, so we draw them all solid and black, except that the internal line is drawn green, and the factor it produces in the formulas is green too. The process we follow is

$$\phi(k_1) + \phi(k_2) \;\to\; \phi(k_3) + \phi(k_4), \qquad k_1 + k_2 = k_3 + k_4. \tag{6}$$

(From here on the momenta are called $k_1, \ldots, k_4$ rather than $p_1, p_2, p_1', p_2'$, to keep primes out of the diagrams.) In every diagram of these notes the incoming particles are on the left, $k_1$ above $k_2$, the outgoing on the right, $k_3$ above $k_4$, and time runs from left to right. In position space the particles start and end at the external points $x_1, \ldots, x_4$ of the correlator. At order $g^2$ there are three connected diagrams, one for each way of splitting the four external points into two pairs that meet at the two vertices $z$ and $w$:




(1): $x_1x_2$ meet at $z$ (2): $x_1x_3$ meet at $z$ (3): $x_1x_4$ meet at $z$

In the first diagram the two incoming particles merge into one, which lives for a while and splits back into two. In the second the upper particle goes from $k_1$ to $k_3$ by emitting a $\phi$, which the lower one absorbs on its way from $k_2$ to $k_4$. The third is the same with the two outgoing particles exchanged (the two lines cross without a dot, so they do not interact). Since all four particles are identical, nothing distinguishes the second diagram from the third except which outgoing momentum is attached where, and both are there.

## 3.2 The first diagram, step by step

We need the propagator and the equation it solves,

$$G_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{-i}{p^2 + m^2 - i\epsilon}\,e^{ip(x-y)}, \qquad (-\partial^2 + m^2)\,G_F(x-y) = -i\,\delta^4(x-y). \tag{7}$$

Start from the value of the first diagram by last week's rules (a propagator on each line, $ig\int d^4z$ at each vertex, symmetry factor 1). It is the part of the correlator with one field per external particle,

$$\begin{aligned}&\langle 0|\,\mathrm{T}\{\phi(x_3)\phi(x_4)\phi(x_1)\phi(x_2)\}\,|0\rangle\Big|_{(1)} \\ &\quad = (ig)^2\int d^4z\,d^4w\;G_F(x_1,z)\,G_F(x_2,z)\;G_F(z,w)\;G_F(w,x_3)\,G_F(w,x_4).\end{aligned} \tag{8}$$

Now apply the LSZ factor (b and c) of each external particle to its own line. For the incoming particle with momentum $k_1$, attached at $z$, Eq. (7) gives

$$i\int d^4x_1\, e^{ik_1x_1}\,(-\partial_1^2+m^2)\,G_F(x_1,z) = i\int d^4x_1\, e^{ik_1x_1}\,(-i\,\delta^4(x_1-z)) = e^{ik_1z}\,: \tag{9}$$

the external line is gone, and the vertex has received a plane wave with the particle's momentum. The same happens on each leg, with $e^{-ik_3w}$ and $e^{-ik_4w}$ for the outgoing ones, and what is left is the part of the S-matrix element that comes from diagram (1):

$$\langle \text{out}:\vec{k}_3\vec{k}_4\,|\,\text{in}:\vec{k}_1\vec{k}_2\rangle\Big|_{(1)} = (ig)^2\int d^4z\,d^4w\; e^{ik_1z}e^{ik_2z}\; e^{-ik_3w}e^{-ik_4w}\; G_F(z,w). \tag{10}$$

Write the internal line in momentum form, Eq. (7), choosing the sign of $p$ so that it flows from $z$ to $w$ as in the diagram, $G_F(z,w) = \int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+m^2-i\epsilon}e^{ip(w-z)}$:

$$(ig)^2\int\frac{d^4p}{(2\pi)^4}\,\frac{-i}{p^2+m^2-i\epsilon}\underbrace{\int d^4z\; e^{i(k_1+k_2-p)z}}_{(2\pi)^4\delta^4(k_1+k_2-p)}\;\underbrace{\int d^4w\; e^{i(p-k_3-k_4)w}}_{(2\pi)^4\delta^4(p-k_3-k_4)}. \tag{11}$$

Each vertex integral has become a delta function: *momentum conservation at a vertex is the vertex integral*. The $p$ integral eats one of them and sets $p = k_1+k_2$, the total momentum, as it must for a particle that is made out of both incoming ones; the other survives,

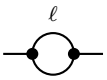
$$\langle \text{out}:\vec{k}_3\vec{k}_4\,|\,\text{in}:\vec{k}_1\vec{k}_2\rangle\Big|_{(1)} = (2\pi)^4\delta^4(k_1+k_2-k_3-k_4)\;\cdot\;(ig)^2\,\frac{-i}{(k_1+k_2)^2+m^2-i\epsilon}. \tag{12}$$

The surviving delta function is the one in Eq. (4). Removing it and adding the other two diagrams, in which the internal line carries $k_1-k_3$ and $k_1-k_4$ instead, gives

$$i\mathcal{M} = (ig)^2\,\frac{-i}{(k_1+k_2)^2+m^2-i\epsilon} + (ig)^2\,\frac{-i}{(k_1-k_3)^2+m^2-i\epsilon} + (ig)^2\,\frac{-i}{(k_1-k_4)^2+m^2-i\epsilon}. \tag{13}$$

### 3.3 The rules in momentum space

Side by side with last week:

| | **position space (correlators)** | **momentum space (amplitudes)** |
|---|---|---|
| $x_i$  | an external point $x_i$ for each field in the correlator | an external line for each incoming or outgoing particle, with its momentum; factor 1 |
| $p$  | every line: $G_F(x,y)$ | every *internal* line with momentum $p$: $-i/(p^2+m^2-i\epsilon)$ |
|  | every vertex: $ig\int d^4z$ | every vertex: $ig$, and momentum is conserved at it |
| $\ell$  $p-\ell$ | — | a momentum not fixed by conservation (a loop, see PS5 problem 3): $\int d^4\ell/(2\pi)^4$ |
| $\sum$ | divide by $S$; add all diagrams | divide by $S$; add all *connected* diagrams: the sum is $i\mathcal{M}$ |

- Symmetry factors are untouched. The vertex factor is $i$ times the coupling times the number of ways to attach the legs: $\frac{1}{3!}g\phi^3$ gives $ig$, the 3! being cancelled by the ways of attaching three identical legs; $g\,\phi\chi\psi$ with three different fields also gives $ig$; $g\,\phi^2\chi$ gives $2ig$; $-\frac{\lambda}{4!}\phi^4$ gives $-i\lambda$.
- If the coupling contains derivatives, Eq. (9) says what to do: the line that brings momentum $k$ into the vertex arrives there as the plane wave $e^{ikz}$, so a $\partial_\mu$ acting on that field just pulls down $ik_\mu$ ($-ik_\mu$ for a momentum flowing out). Each derivative in the vertex becomes a factor of the momentum of the line it acts on.

### 3.4 Mandelstam variables

The three denominators in Eq. (13) depend on the external momenta only through Lorentz invariant squares, so $\mathcal{M}$ can be written without choosing a frame. For a $2 \to 2$ process three such squares are the *Mandelstam variables*

$$s = -(k_1 + k_2)^2, \qquad t = -(k_1 - k_3)^2, \qquad u = -(k_1 - k_4)^2, \tag{14}$$

with the convention that $t$ pairs the upper incoming with the upper outgoing particle. Expanding the squares, using $k_1 + k_2 = k_3 + k_4$ and $k_i^2 = -m^2$, they satisfy

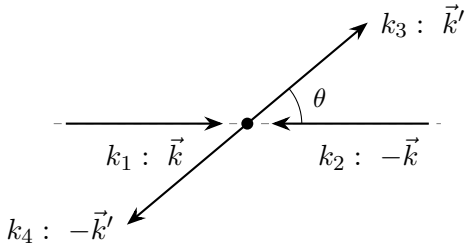
$$s + t + u = 4m^2, \tag{15}$$

so only two are independent. In terms of them Eq. (13) reads

$$\boxed{\mathcal{M}(\phi\phi \to \phi\phi) = g^2 \left[\frac{1}{m^2 - s} + \frac{1}{m^2 - t} + \frac{1}{m^2 - u}\right].}$$

Each term says "the internal particle would like to be real": the closer $s$ is to $m^2$, the closer the intermediate particle is to its mass shell, the longer it lives, the larger the amplitude. Each diagram is named after the variable in its denominator: (1) is the *s-channel*, (2) the *t-channel* and (3) the *u-channel*.

**Centre-of-mass frame.** The frame in which $\vec{k}_1 + \vec{k}_2 = 0$: the two particles come in along a line with opposite momenta $\pm\vec{k}$; they have equal masses, so energy conservation sends them out along another line with opposite momenta $\pm\vec{k}'$ of the same size, $|\vec{k}'| = |\vec{k}|$ (the collision is *elastic*), rotated by the scattering angle $\theta$:




Taking $\vec{k}$ along the $\hat{z}$ axis, the four momenta are

$$\begin{aligned} k_1 &= (E,\, 0, 0, |\vec{k}|), & k_3 &= (E,\, |\vec{k}|\sin\theta, 0, |\vec{k}|\cos\theta), \\ k_2 &= (E,\, 0, 0, -|\vec{k}|), & k_4 &= (E,\, -|\vec{k}|\sin\theta, 0, -|\vec{k}|\cos\theta), \end{aligned} \tag{16}$$

all with the same energy $E = \sqrt{|\vec{k}|^2 + m^2}$. The total energy is $\sqrt{s} = 2E$, which fixes $|\vec{k}|$ in terms of $s$, and

$$s = 4(|\vec{k}|^2 + m^2), \qquad t = -2|\vec{k}|^2(1 - \cos\theta), \qquad u = -2|\vec{k}|^2(1 + \cos\theta). \tag{17}$$

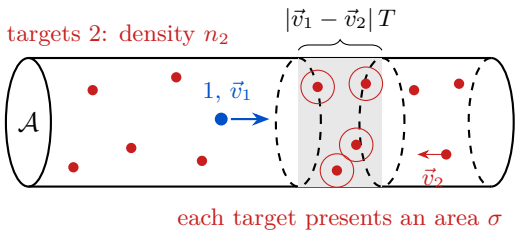
Since $s \geq 4m^2 > m^2$ while $t, u \leq 0 < m^2$, none of the three denominators ever vanishes: $\mathcal{M}$ is finite and real. The $t$-term is largest for forward scattering, $\theta \to 0$, where $t \to 0$, the $u$-term for backward scattering, $\theta \to \pi$, and the $s$-term does not depend on the angle.

# 4 Cross sections and decay rates

## 4.1 What a cross section is

We have $\mathcal{M}$. The real observable in the experiment is not just $\mathcal{M}$ or $|\mathcal{M}|^2$, it is usually either the cross section or the decay rate. Both are defined by what a detector *counts*: for a beam hitting a target, or for unstable particles decaying.

As an intuitive derivation of the cross section consider the following. An experiment never collides just two particles: it collides *beams*, bunches of some $10^{11}$ particles each, or sends one beam into a block of target material. Call the beam particles "1" and the target particles "2", with number densities $n_1$, $n_2$ and velocities $\vec{v}_1$, $\vec{v}_2$, and let the experiment run for a time $T$. The *probability* of scattering is the fraction of incoming particles that scatter, $P = N_{\text{scat}}/N_{\text{inc}}$. To find it, follow a single beam particle through a cylinder filled with targets:



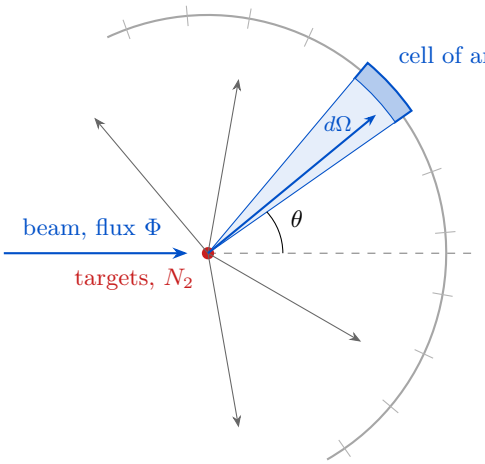

In the time $T$ it moves a distance $|\vec{v}_1 - \vec{v}_2|\,T$ relative to the targets, so it sweeps through a slab of the cylinder of base area $\mathcal{A}$ that contains $n_2|\vec{v}_1 - \vec{v}_2|\,\mathcal{A}\,T$ targets (we take the slab thin enough that the targets do not hide each other). Suppose each target presents an effective area $\sigma$: a beam particle that passes through that area scatters, one that misses it does not. The targets cover an area $\sigma\, n_2|\vec{v}_1 - \vec{v}_2|\,\mathcal{A}\,T$ of the base $\mathcal{A}$, and the beam particle can be anywhere on the base, so its probability to scatter is

$$P = \sigma\, n_2\, |\vec{v}_1 - \vec{v}_2|\, T. \tag{18}$$

This *defines* $\sigma$ for any process, hard sphere or not: $n_2$, $|\vec{v}_1 - \vec{v}_2|$ and $T$ describe only the set-up, and all the physics is in $\sigma$. A whole beam of $N_{\text{inc}}$ particles gives $N_{\text{scat}} = P\, N_{\text{inc}}$; in terms of the *flux* $\Phi = n_1|\vec{v}_1 - \vec{v}_2|$ (beam particles crossing a unit area per unit time, as seen from a target) and the number of targets $N_2$, this is $N_{\text{scat}} = \Phi\, N_2\, T\, \sigma$. The product $L = \Phi N_2$ is the *luminosity*, a name borrowed from optics: like the brightness of a lamp, it says how many particles per unit area per unit time the machine delivers.[3]

The detector does more than count: it is split into cells, and each cell counts the particles that fly into it.

[3] Cross sections are quoted in barns, $1\,\text{b} = 10^{-24}\,\text{cm}^2$; in natural units $1\,\text{GeV}^{-2} = 0.389\,\text{mb}$. Two protons at the LHC interact inelastically with $\sigma \approx 80\,\text{mb}$ and make a Higgs boson with $\sigma \approx 50\,\text{pb}$, one time in $10^9$; with $L \approx 2 \times 10^{34}\,\text{cm}^{-2}\text{s}^{-1}$ that is about one Higgs per second.




black and blue: all scattered particles
blue: those that land in the cell

A cell of area $dA$ at distance $r$ from the target, in the direction $\theta$ from the beam, covers the solid angle $d\Omega = dA/r^2$ ($d\Omega = \sin\theta\, d\theta\, d\phi$ in spherical coordinates, and the whole sphere is $\int d\Omega = 4\pi$). It catches only the particles scattered into that direction, and Eq. (18) holds for them with $\sigma$ replaced by the part of the cross section that sends particles into the cell. That part is $d\sigma$, and the *differential cross section* $d\sigma/d\Omega$ is $d\sigma$ per unit solid angle:

$$dP = d\sigma\; n_2\, |\vec{v}_1 - \vec{v}_2|\, T, \qquad \sigma = \int d\Omega\, \frac{d\sigma}{d\Omega}. \tag{19}$$

## 4.2 The cross section and the decay rate from $\mathcal{M}$

To compute $\sigma$, put one beam particle and one target in a box of volume $V$ for the time $T$. Then $n_2 = 1/V$, and for one detector cell Eq. (18) reads $dP = d\sigma\, |\vec{v}_1 - \vec{v}_2|\, T/V$, so

$$d\sigma = \frac{V}{T}\, \frac{1}{|\vec{v}_1 - \vec{v}_2|}\, dP. \tag{20}$$

Now $dP$ is a quantum-mechanical probability: the probability that the outgoing particles land in the cell. The cell in the figure catches the particles whose momentum $\vec{k}'$ points into its solid angle; writing $\vec{k}'$ as a size and a direction, that region of momentum space is $d^3k' = |\vec{k}'|^2\, d|\vec{k}'|\, d\Omega$ (energy conservation will fix $|\vec{k}'|$, leaving only the direction $d\Omega$). To count the states in it: in a box of volume $V = L^3$ a plane wave $e^{i\vec{k}\cdot\vec{x}}$ has to fit into the box, so $\vec{k} = 2\pi\vec{n}/L$ with $\vec{n}$ a vector of integers, and each allowed momentum takes up a volume $(2\pi)^3/V$ of momentum space. The probability is the normalized amplitude squared times the number of final states, for each of the $n$ outgoing particles (from here on $n$ counts the outgoing particles):

$$\begin{aligned} dP &= \frac{|\langle f|i\rangle|^2}{\langle f|f\rangle\, \langle i|i\rangle} \prod_{j=1}^{n} \frac{V\, d^3k'_j}{(2\pi)^3} \\ &= \underbrace{VT\,(2\pi)^4\delta^4(\cdots)\,|\mathcal{M}|^2}_{\substack{|\langle f|i\rangle|^2,\ \text{Eq. (4)},\\ (2\pi)^4\delta^4(0) = VT}}\ \underbrace{\frac{1}{(2E_1V)(2E_2V)\prod_j 2E'_jV}}_{\substack{1/\langle i|i\rangle\langle f|f\rangle,\\ \langle\vec{k}|\vec{k}\rangle = 2EV}}\ \underbrace{\prod_j \frac{V\, d^3k'_j}{(2\pi)^3}}_{\text{states in the cell}} \\ &= \frac{T}{V}\, \frac{|\mathcal{M}|^2}{4E_1E_2}\, d\mathrm{LIPS}_n, \end{aligned} \tag{21}$$

with $d\mathrm{LIPS}_n = (2\pi)^4\delta^4(\cdots)\prod_j d^3k'_j/((2\pi)^3\, 2E'_j)$. Every $V$ of the final states cancels, and the $T/V$ left over exactly cancels the time and volume factors in Eq. (20).

**Decay.** No beam. Put $N$ unstable particles at rest and count the number $N_{\rm dec}$ that decay in a time $T$, short compared with their lifetime. The fraction that decays is proportional to $T$, and the constant of proportionality is the decay rate,

$$P = \frac{N_{\rm dec}}{N} = \Gamma\, T : \tag{22}$$

the probability per unit time that one particle decays. If several final states are possible their rates add, $\Gamma = \sum_i \Gamma_i$, with $\Gamma_i/\Gamma$ the *branching ratio* into channel $i$. With one particle in the box, $\langle i|i\rangle = 2E_1 V$, the same steps give $dP = T\,|\mathcal{M}|^2\, d\mathrm{LIPS}_n/(2E_1)$, and $d\Gamma = dP/T$. Together:

$$\boxed{\begin{aligned} d\sigma &= \frac{|\mathcal{M}|^2}{4E_1E_2\,|\vec v_1 - \vec v_2|}\; d\mathrm{LIPS}_n, \qquad\qquad d\Gamma = \frac{|\mathcal{M}|^2}{2E_1}\; d\mathrm{LIPS}_n, \\ d\mathrm{LIPS}_n &= (2\pi)^4\delta^4\Big(\textstyle\sum_j k'_j - \sum_i k_i\Big)\prod_{j=1}^n \frac{d^3k'_j}{(2\pi)^3\,2E'_j}. \end{aligned}} \tag{23}$$

Read them factor by factor. $|\mathcal{M}|^2$ is the probability of the transition, the only factor that needs diagrams. $d\mathrm{LIPS}_n$, the Lorentz-invariant phase space, counts *how many final states there are*. The $1/2E$ per incoming particle undoes the normalization of our states, and $1/|\vec v_1 - \vec v_2|$ is what is left of the flux; a decay has no beam and no such factor.

### 4.3 Two-body phase space

Almost every computation this term ends with two particles in the final state, so let us do $n = 2$, calling the outgoing momenta $k_3$, $k_4$ and their masses $m_3$, $m_4$. Start from $d\mathrm{LIPS}_n$ in Eq. (23), with $P$ the total incoming momentum:

$$d\mathrm{LIPS}_2 = \int \frac{d^3\vec k_3}{(2\pi)^3}\,\frac{1}{2E_3}\int \frac{d^3\vec k_4}{(2\pi)^3}\,\frac{1}{2E_4}\,(2\pi)^4\delta^3(\vec P - \vec k_3 - \vec k_4)\,\delta(P^0 - E_3 - E_4), \tag{24}$$

with $E_j = \sqrt{|\vec k_j|^2 + m_j^2}$. Now go to the centre-of-mass frame:

$$\boxed{\vec P = 0 \quad \text{and} \quad P^0 = \sqrt{s}}$$

$$d\mathrm{LIPS}_2 = \int \frac{d^3\vec k_3}{(2\pi)^3}\,\frac{1}{2E_3}\int \frac{d^3\vec k_4}{(2\pi)^3}\,\frac{1}{2E_4}\,(2\pi)^4\delta^3(-\vec k_3 - \vec k_4)\,\delta(\sqrt{s} - E_3 - E_4). \tag{25}$$

Integrating over $\vec k_4$:

$$\boxed{\vec k_4 \to -\vec k_3}$$

$$d\mathrm{LIPS}_2 = \int \frac{|\vec k_3|^2\, d|\vec k_3|\, d\Omega}{4(2\pi)^2}\,\frac{1}{E_3E_4}\,\delta\Big(\sqrt{s} - \sqrt{|\vec k_3|^2 + m_3^2} - \sqrt{|\vec k_3|^2 + m_4^2}\Big), \tag{26}$$

with $E_4 = \sqrt{|\vec k_3|^2 + m_4^2}$ now. The delta function appearing in the integral can be evaluated using the identity

$$\delta(g(x)) = \sum_{x^*} \frac{\delta(x - x^*)}{|g'(x^*)|}, \tag{27}$$

where in our case $x = |\vec{k}_3|$ and $g(x) = \sqrt{s} - \sqrt{x^2+m_3^2} - \sqrt{x^2+m_4^2}$. The root of this equation is

$$|\vec{k}_3|^* \equiv |\vec{k}'| = \frac{\sqrt{(s-(m_3+m_4)^2)(s-(m_3-m_4)^2)}}{2\sqrt{s}}\,, \tag{28}$$

the momentum that makes the two energies add up to $\sqrt{s}$, and $|g'(x^*)| = |\vec{k}'|/E_3 + |\vec{k}'|/E_4 = |\vec{k}'|\sqrt{s}/(E_3E_4)$, so the delta function reduces to

$$\delta(\sqrt{s} - E_3 - E_4) = \delta(|\vec{k}_3| - |\vec{k}'|)\,\frac{E_3E_4}{|\vec{k}'|\sqrt{s}}\,. \tag{29}$$

Performing the integration over $|\vec{k}_3|$ we obtain:

$$\boxed{|\vec{k}_3| \to |\vec{k}'|}$$

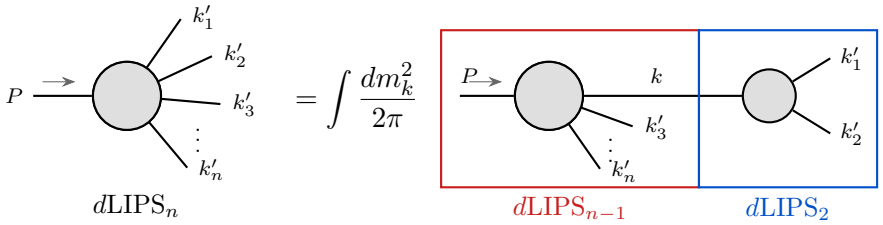
$$d\text{LIPS}_2 = \int \frac{|\vec{k}_3|^2\, d|\vec{k}_3|\, d\Omega}{4(2\pi)^2}\,\frac{1}{E_3E_4}\,\frac{E_3E_4}{|\vec{k}'|\sqrt{s}}\,\delta(|\vec{k}_3| - |\vec{k}'|) = \int d\Omega\,\frac{|\vec{k}'|}{16\pi^2\sqrt{s}}\,, \tag{30}$$

that is, per element of solid angle in the centre-of-mass frame,

$$\boxed{d\text{LIPS}_2 = \frac{|\vec{k}'|}{16\pi^2\sqrt{s}}\,d\Omega_{\text{cm}}} \tag{31}$$

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**Note: more particles.** An $n$-body phase space can always be split into two-body ones:




Inserting $1 = \int \frac{dm_k^2}{2\pi} \int \frac{d^3\vec{k}}{(2\pi)^3\,2E_k}\,(2\pi)^4\delta^4(k - k_1' - k_2')$, with $E_k = \sqrt{|\vec{k}|^2 + m_k^2}$, into $d\text{LIPS}_n$ of Eq. (23) groups $k_1', k_2'$ into one particle $k$ of variable mass $m_k$: $P$ first decays into $k$ and the rest ($d\text{LIPS}_{n-1}$), then $k \to k_1'k_2'$ ($d\text{LIPS}_2$). Each factor is Lorentz invariant, so it can be evaluated in its own rest frame.

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### 4.4 Our process: $2 \to 2$

A $2 \to 2$ cross section is Eq. (23) with two particles in the final state,

$$d\sigma = \frac{|\mathcal{M}|^2}{4E_1E_2\,|\vec{v}_1 - \vec{v}_2|}\,d\text{LIPS}_2. \tag{32}$$

The phase space was computed in the centre-of-mass frame, so we write the flux factor there too. A particle of momentum $\vec{k}$ and energy $E = \sqrt{|\vec{k}|^2 + m^2}$ moves with velocity $\vec{v} = \vec{k}/E$ (from $\vec{k} = \gamma m\vec{v}$ and $E = \gamma m$). There $\vec{k}_1 = \vec{k}$ and $\vec{k}_2 = -\vec{k}$, so the velocities are opposite and

$$|\vec{v}_1 - \vec{v}_2| = \frac{|\vec{k}|}{E_1} + \frac{|\vec{k}|}{E_2} = \frac{|\vec{k}|\,(E_1+E_2)}{E_1E_2} = \frac{|\vec{k}|\sqrt{s}}{E_1E_2}, \qquad \text{so} \qquad 4E_1E_2|\vec{v}_1 - \vec{v}_2| = 4|\vec{k}|\sqrt{s}, \tag{33}$$

with $\sqrt{s} = E_1 + E_2$ the total energy. Multiplying Eq. (31) by $|\mathcal{M}|^2$ and dividing by this flux factor gives

$$\boxed{\frac{d\sigma}{d\Omega_{\rm cm}} = \frac{|\mathcal{M}|^2}{64\pi^2 s}\,\frac{|\vec{k}'|}{|\vec{k}|}} \tag{34}$$

for any $2 \to 2$ process, with $|\vec{k}|$ and $|\vec{k}'|$ the sizes of the incoming and outgoing momenta. For $\phi\phi \to \phi\phi$ the collision is elastic, $|\vec{k}'| = |\vec{k}|$, and Eqs. (17), (34) give

$$\frac{d\sigma}{d\Omega_{\rm cm}} = \frac{g^4}{64\pi^2 s}\left[\frac{1}{m^2 - s} + \frac{1}{m^2 - t} + \frac{1}{m^2 - u}\right]^2 . \tag{35}$$

The solid angle is $d\Omega_{\rm cm} = \sin\theta\, d\theta\, d\phi$, and nothing depends on the azimuth $\phi$. The two outgoing particles are identical, so "one at $\theta$, the other at $\pi - \theta$" and its exchange are the *same* final state, which the full $\int d\Omega$ counts twice; the total cross section therefore gets a $\frac{1}{2}$ (the differential one does not):

$$\sigma = \frac{1}{2}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, \frac{d\sigma}{d\Omega_{\rm cm}} = \pi \int_{-1}^{1} d\cos\theta\, \frac{d\sigma}{d\Omega_{\rm cm}}, \tag{36}$$

with $\theta$ entering only through $t = -2|\vec{k}|^2(1 - \cos\theta)$ and $u = -2|\vec{k}|^2(1 + \cos\theta)$, Eq. (17).