Section 7: Regularization and Renormalization

Wednesday, October 21st 2026

Srednicki Ch. 13, 14, 27; Schwartz Ch. 15, 18–19, App. B; Peskin & Schroeder §6.3, §7.1, §10.2, App. A.4

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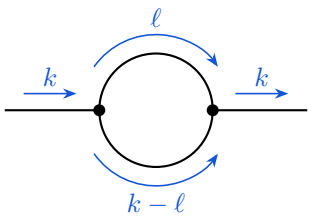
1 Intro

We work with ϕ^3 theory in d spacetime dimensions,

$$\mathcal{L} = -\tfrac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{6}\,g\,\phi^3 + Y\phi, \tag{1}$$

with Y tuned so that $\langle\Omega|\phi|\Omega\rangle = 0$, as in lecture; then the tadpole diagrams cancel. The lecture's Lagrangian also carries factors Z in front of each term; for now we set them to 1.

In lecture we computed our first diagram with a closed loop in this theory: the one-loop correction to the propagator,



In a tree diagram, momentum conservation at the vertices fixes every internal momentum. Here it does not: whatever momentum ℓ runs along the upper line, the lower line carries $k-\ell$, and nothing fixes ℓ. So we integrate over it. With the external lines removed, the Feynman rules give for the value of this diagram

$$i\Pi_{\text{loop}}(k^2) = \frac{g^2}{2}\int\frac{d^d\ell}{(2\pi)^d}\,\frac{1}{(\ell^2+m^2-i\varepsilon)\,((\ell-k)^2+m^2-i\varepsilon)}\,, \tag{2}$$

and this integral is infinite.

A first look at the problem. Take zero external momentum, $k=0$:

$$\Pi_{\text{loop}}(0) = \frac{g^2}{2}\int\frac{d^d\ell}{(2\pi)^d}\,\frac{1}{(\ell^2+m^2)^2} \qquad \text{(in Euclidean space)}. \tag{3}$$

The integral adds up the contributions of all the momenta ℓ the virtual particles in the loop can carry, including arbitrarily large ones. To find out whether that sum makes sense, keep only $|\ell| < \Lambda$ for some large momentum Λ and watch what happens as Λ is pushed up. Such a Λ is called a *cutoff*: a probe we put in by hand, not part of the theory. In spherical coordinates, $d^d\ell = \Omega_d\, \ell^{\,d-1} d\ell$ with $\ell \equiv |\ell|$ and Ω_d the total solid angle. For $\ell \gg m$ the integrand is $\approx \ell^{\,d-5}$, so

$$\Pi_{\text{loop}}(0) = \frac{g^2}{2}\,\frac{\Omega_d}{(2\pi)^d}\int_0^\Lambda d\ell\;\frac{\ell^{\,d-1}}{(\ell^2+m^2)^2} \;\sim\; g^2\int^\Lambda d\ell\;\ell^{\,d-5} \;\sim\; \begin{cases} g^2\Lambda^{d-4}, & d>4,\\ g^2\ln\Lambda, & d=4, \end{cases} \tag{4}$$

which grows without bound as we push $\Lambda \to \infty$.

The infinity does not mean the theory is broken. It means that we have been treating the mass and the coupling written in the Lagrangian as if they were the numbers an experiment measures, and the loop is telling us that they cannot be. The cure comes in two moves:

First regularize, then renormalize.

- **Regularize**: modify the theory at very high momenta by some (arbitrary) rule, so that the loop integral becomes finite and the infinity turns into an explicit term like $\ln\Lambda$, or a pole $1/\epsilon$.
- **Renormalize**: only once the pole is clearly separated, ask what is actually measured and rewrite the theory in terms of those quantities. The pole is then swept into the parameters of the Lagrangian, which nobody ever measured, and every prediction comes out finite and independent of the regulator.

Why six dimensions? Since the bubble grows like Λ^{d-4}, the dimension decides what the infinity looks like. In $d = 4$ it is only a $\ln\Lambda$, and it cannot depend on k (it has dimension zero), so it would only shift the mass. In $d = 6$ the infinity has mass dimension 2, so it can contain three different structures,

$$\Lambda^2, \qquad m^2\ln\Lambda, \qquad k^2\ln\Lambda. \tag{5}$$

The first two shift the mass. The k^2 term can only be absorbed by changing the coefficient of $\partial_\mu\phi\,\partial^\mu\phi$, i.e. the normalization of the field. So in six dimensions a single one-loop diagram already needs both a mass and a field renormalization: the whole pipeline is visible in one diagram. (A bonus: in $d = 6$ the coupling g is dimensionless, $[g] = 3 - \frac{d}{2} = 0$, just like the couplings of the Standard Model in four dimensions, so ϕ^3 in six dimensions is a good toy model for the real thing.)

2 How to extract the divergence: regularization schemes

From now on $d = 6$, where the bubble diverges like Λ^2. A regulator changes the theory at momenta far above anything we probe, where we do not know the physics anyway. Three standard choices:

Momentum cutoff. Keep only loop momenta $|\ell| < \Lambda$ (in Euclidean space).

Useful: the most intuitive, Λ is literally the highest momentum we trust (the language of effective field theories and the lattice).

But: it is not invariant under shifts $\ell \to \ell + k$, so the answer depends on how we route the momenta, and it breaks gauge invariance.

Pauli–Villars. Subtract from every propagator that of a fictitious heavy particle, $\frac{1}{\ell^2+m^2} \to \frac{1}{\ell^2+m^2} - \frac{1}{\ell^2+M^2}$, which falls like M^2/ℓ^4 at large ℓ, so the bubble converges.

Useful: the regulated theory is still Lorentz invariant (and gauge invariant in QED).

But: clumsy beyond simple theories.

Dimensional regularization. Compute in $d < 4$, where the integral converges, and continue to $d = 6 - \epsilon$; the divergence shows up as a pole $1/\epsilon$.

Useful: it keeps every symmetry, has no power divergences, and all integrals reduce to one master formula, which is why it is the standard choice.

But: no direct physical picture.

All regulators give the same physics. They disagree only about the divergent pieces (Λ^2 and $\ln\Lambda$, M^2 and $\ln M$, or $1/\epsilon$), which are polynomials in m^2 and k^2. Part II absorbs exactly such polynomials into the parameters of the Lagrangian, so every physical prediction comes out regulator independent.

From here on we use dimensional regularization.

3 Part I: regularize

Step 1 • From the diagram to an integral

Route the momentum as in the diagram: k splits at the first vertex into ℓ on the upper line and $k-\ell$ on the lower line, and recombines at the second. Nothing fixes ℓ, so it is integrated over. The Feynman rules give ig per vertex, $-i/(p^2+m^2-i\varepsilon)$ per internal line, $\int d^6\ell/(2\pi)^6$ for the loop, the symmetry factor $\frac{1}{2}$:

$$\begin{aligned} i\Pi_{\text{loop}}(k^2) &= \frac{1}{2}(ig)^2 \int \frac{d^6\ell}{(2\pi)^6} \frac{-i}{\ell^2+m^2-i\varepsilon} \frac{-i}{(k-\ell)^2+m^2-i\varepsilon} \\ &= \frac{g^2}{2} \int \frac{d^6\ell}{(2\pi)^6} \frac{1}{(\ell^2+m^2-i\varepsilon)\left((\ell-k)^2+m^2-i\varepsilon\right)}. \end{aligned} \tag{6}$$

Step 2 • Regulate: dimensional regularization

The rule. Replace $d^6\ell/(2\pi)^6$ by $d^d\ell/(2\pi)^d$ and keep d as a variable. So we can set

$$d = 6-\epsilon, \qquad \epsilon \to 0^+ \text{ at the very end.} \tag{7}$$

Keep the coupling dimensionless. In $d=6-\epsilon$ dimensions $[g] = 3-\frac{d}{2}$ gives

$$[g] = 3 - \frac{6-\epsilon}{2} = \frac{\epsilon}{2}. \tag{8}$$

We want g to stay the same dimensionless number as in $d=6$, so we pull out the dimension with an arbitrary mass scale $\tilde{\mu}$:

$$g \;\to\; g\,\tilde{\mu}^{\,\epsilon/2}.$$

The scale $\tilde{\mu}$ is part of the regulator, like Λ in a cutoff, and has to drop out of physical answers. Without it we would soon be taking logarithms of dimensionful quantities. The regulated bubble is

$$i\Pi_{\text{loop}}(k^2) = \frac{g^2\tilde{\mu}^{\,\epsilon}}{2} \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2+m^2-i\varepsilon)\left((\ell-k)^2+m^2-i\varepsilon\right)}. \tag{9}$$

Step 3 • Feynman parameters

For two denominators A and B that do not vanish (and have imaginary parts of the same sign, so the combination below never vanishes either),

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{\left[xA+(1-x)B\right]^2}. \tag{10}$$

Take $A=(\ell-k)^2+m^2-i\varepsilon$, $B=\ell^2+m^2-i\varepsilon$.

Expand the combination and complete the square in ℓ:

$$\begin{aligned} xA+(1-x)B &= x(\ell^2-2\ell\cdot k+k^2)+(1-x)\,\ell^2+m^2-i\varepsilon \\ &= \ell^2-2x\,\ell\cdot k+xk^2+m^2-i\varepsilon \\ &= (\ell-xk)^2-x^2k^2+xk^2+m^2-i\varepsilon \\ &= (\ell-xk)^2+D, \end{aligned}$$

where in the third line we used $(\ell-xk)^2=\ell^2-2x\ell\cdot k+x^2k^2$, and

$$D \equiv x(1-x)\,k^2+m^2-i\varepsilon. \tag{11}$$

$$i\Pi_{\text{loop}} = \frac{g^2\tilde{\mu}^{\,\epsilon}}{2} \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left[(\ell-xk)^2+D\right]^2}. \tag{12}$$

Step 4 • Shift the loop momentum and Wick-rotate

Shift. Change variables to $q = \ell - xk$. At fixed x this change of variables is a translation, so $d^d q = d^d \ell$ and the range is still all of $\mathbb{R}^d$:

$$i\Pi_{\text{loop}} = \frac{g^2 \tilde{\mu}^{\,\epsilon}}{2} \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \, \frac{1}{(q^2 + D)^2}. \tag{13}$$

Where the poles are. Do the q^0 integral first, at fixed $\vec{q}$. The integrand blows up where $q^2 + D = 0$, i.e. where

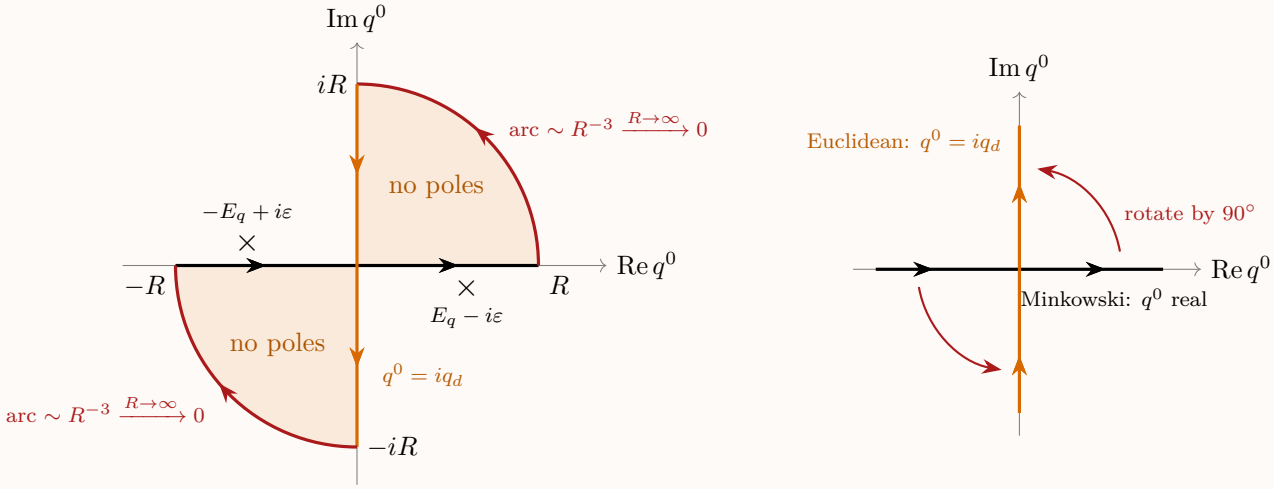
$$(q^0)^2 = \vec{q}^{\,2} + D = E_q^2 - i\varepsilon, \qquad E_q \equiv \sqrt{\vec{q}^{\,2} + x(1-x)k^2 + m^2} \tag{14}$$

i.e. at

$$q^0 = \pm\sqrt{E_q^2 - i\varepsilon} = \pm\Big(E_q - \frac{i\varepsilon}{2E_q}\Big) \;\to\; \pm(E_q - i\varepsilon) \tag{15}$$

(the positive factor $1/2E_q$ can be absorbed into the infinitesimal ε): one pole just below the positive real axis, the other just above the negative real axis.

Rotate. Close the contour as in the diagram: along the real axis from $-R$ to R, along the arc up to iR, down the imaginary axis to $-iR$, and along the arc back to $-R$.



In short (right diagram), the contour along the real axis can be rotated counterclockwise onto the imaginary axis without crossing a pole. In detail (left diagram), the closed contour encloses only the shaded quarter discs, which contain no poles, so by Cauchy's theorem the integral around it vanishes. On the arcs the integrand $F(q^0)$ falls like $|q^0|^{-4}$ while their length grows like R, so they contribute $\sim R^{-3}$. On the imaginary axis set $q^0 = iq_d$ with q_d real, so $dq^0 = i\,dq_d$; it is traversed downwards, hence the minus sign:

$$\begin{aligned} 0 = \oint dq^0 \, F(q^0) &= \int_{-R}^{R} dq^0 \, F(q^0) + \int_{\text{arcs}} dq^0 \, F(q^0) - i \int_{-R}^{R} dq_d \, F(iq_d) \\ &\xrightarrow{R\to\infty} \qquad \int_{-\infty}^{\infty} dq^0 \, F(q^0) = i \int_{-\infty}^{\infty} dq_d \, F(iq_d). \end{aligned} \tag{16}$$

The Minkowski square turns into a Euclidean one,

$$q^2 = -(iq_d)^2 + \vec{q}^{\,2} = q_d^2 + \vec{q}^{\,2} \equiv \bar{q}^{\,2}, \qquad \bar{q} = (q_1, \ldots, q_{d-1}, q_d), \qquad d^d q = i\, d^d \bar{q}. \tag{17}$$

Inserting this, the factor i cancels the i on the left-hand side:

$$\Pi_{\text{loop}}(k^2) = \frac{g^2 \tilde{\mu}^{\,\epsilon}}{2} \int_0^1 dx \int \frac{d^d \bar{q}}{(2\pi)^d} \, \frac{1}{(\bar{q}^{\,2} + D)^2}. \tag{18}$$

Step 5 • The master integral

We use the formula from lecture,

$$I(a,b,d) \equiv \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{(\bar{q}^2)^a}{(\bar{q}^2+D)^b} = \frac{\Gamma(b-a-\frac{d}{2})\,\Gamma(a+\frac{d}{2})}{(4\pi)^{d/2}\,\Gamma(b)\,\Gamma(\frac{d}{2})}\, D^{-(b-a-\frac{d}{2})}. \tag{19}$$

The integral converges for $0 < a + \frac{d}{2} < b$; outside that range the right-hand side is the analytic continuation in d, which in dimensional regularization *is* the definition of the integral.

Apply it with $a = 0$, $b = 2$. Then $\Gamma(a+\frac{d}{2}) = \Gamma(\frac{d}{2})$ cancels against the denominator and $\Gamma(2) = 1$:

$$I(0,2,d) = \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\, D^{\frac{d}{2}-2}. \tag{20}$$

Inserting this in Π_{loop} gives

$$\Pi_{\text{loop}}(k^2) = \frac{g^2\tilde{\mu}^{\,\epsilon}}{2}\,\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dx\; D^{\frac{d}{2}-2}. \tag{21}$$

Step 6 • Expand as $\epsilon \to 0$

Set $d = 6 - \epsilon$, so

$$\Pi_{\text{loop}}(k^2) = \frac{g^2\tilde{\mu}^{\,\epsilon}}{2}\,\frac{\Gamma(-1+\frac{\epsilon}{2})}{(4\pi)^{3-\epsilon/2}} \int_0^1 dx\; D^{1-\frac{\epsilon}{2}}.$$

We need two expansions.

(i) The Γ-function. Use $\Gamma(z+1) = z\,\Gamma(z)$ twice to move the argument to the neighbourhood of 1:

$$\Gamma(-1+z) = \frac{\Gamma(z)}{-1+z} = \frac{\Gamma(1+z)}{z\,(z-1)}. \tag{22}$$

Near $z = 0$, $\Gamma(1+z) = 1 - \gamma_E z + \mathcal{O}(z^2)$, where $\gamma_E = -\Gamma'(1) = 0.5772\ldots$ is the Euler–Mascheroni constant, and $\frac{1}{z-1} = -(1+z+z^2+\cdots)$. Multiplying out, and with $z = \epsilon/2$:

$$\Gamma\Big(-1+\frac{\epsilon}{2}\Big) = -\frac{2}{\epsilon} + \gamma_E - 1 + \mathcal{O}(\epsilon). \tag{23}$$

(ii) The powers. With $(4\pi)^{-d/2} = (4\pi)^{-3}(4\pi)^{\epsilon/2}$, $D^{\frac{d}{2}-2} = D\cdot D^{-\epsilon/2}$ and $\tilde{\mu}^{\,\epsilon} = (\tilde{\mu}^{\,2})^{\epsilon/2}$,

$$\tilde{\mu}^{\,\epsilon}\,(4\pi)^{-d/2}\,D^{\frac{d}{2}-2} = \frac{D}{(4\pi)^3}\;\underbrace{\Big(\frac{4\pi\tilde{\mu}^{\,2}}{D}\Big)^{\epsilon/2}}_{1+\frac{\epsilon}{2}\ln\frac{4\pi\tilde{\mu}^{\,2}}{D}+\mathcal{O}(\epsilon^2)} \tag{24}$$

The product. Inserting both expansions, Π_{loop} becomes

$$\Pi_{\text{loop}} = \frac{g^2}{2(4\pi)^3}\int_0^1 dx\, D\Big[-\frac{2}{\epsilon}+\gamma_E-1+\mathcal{O}(\epsilon)\Big]\Big[1+\frac{\epsilon}{2}\ln\frac{4\pi\tilde{\mu}^{\,2}}{D}+\mathcal{O}(\epsilon^2)\Big]. \tag{25}$$

Multiplying the brackets term by term gives

$$\begin{aligned}\Pi_{\text{loop}} &= -\frac{g^2}{2(4\pi)^3}\int_0^1 dx\, D\Big[\frac{2}{\epsilon}+1-\gamma_E+\ln\frac{4\pi\tilde{\mu}^{\,2}}{D}\Big]+\mathcal{O}(\epsilon)\\ &= -\frac{g^2}{2(4\pi)^3}\int_0^1 dx\, D\Big[\frac{2}{\epsilon}+1+\ln\frac{4\pi e^{-\gamma_E}\tilde{\mu}^{\,2}}{D}\Big]+\mathcal{O}(\epsilon).\end{aligned} \tag{26}$$

The constants $-\gamma_E$ and $\ln 4\pi$ come from the Γ-function and from $(4\pi)^{-d/2}$, so they accompany the pole in every loop. Naming the combination

$$\mu^2 \equiv 4\pi e^{-\gamma_E}\,\tilde{\mu}^{\,2}, \tag{27}$$

we arrive at

$$\Pi_{\text{loop}}(k^2) = -\frac{g^2}{2(4\pi)^3}\int_0^1 dx\; D\left[\frac{2}{\epsilon} + 1 + \ln\frac{\mu^2}{D}\right], \qquad D = x(1-x)k^2 + m^2 - i\varepsilon. \tag{28}$$

The pole. The x-integral of D is elementary,

$$\int_0^1 dx\, D = k^2\int_0^1 dx\, x(1-x) + m^2 = k^2\Big(\frac{1}{2} - \frac{1}{3}\Big) + m^2 = \frac{k^2}{6} + m^2, \tag{29}$$

so

$$\Pi_{\text{loop}}(k^2) = -\frac{g^2}{(4\pi)^3\,\epsilon}\Big(\frac{k^2}{6} + m^2\Big) \;+\; (\text{finite as } \epsilon \to 0). \tag{30}$$

The infinite part is linear in k^2 and m^2, as we expected from the three structures Λ^2, $m^2\ln\Lambda$, $k^2\ln\Lambda$.

4 Part II: renormalize

An infinity. Did we do something wrong? Part I ended with

$$\Pi_{\text{loop}}(k^2) = -\frac{g^2}{(4\pi)^3\,\epsilon}\Big(\frac{k^2}{6} + m^2\Big) + \text{finite}, \tag{31}$$

and when we remove the regulator, $\epsilon \to 0$, this blows up. A correction to something we can measure cannot be infinite, so did we make a mistake?

No! Every step of the calculation was fine. The mistake is in how we read the Lagrangian. We have been treating m and g as if they were the mass and the coupling an experiment measures, but nothing ever told us that: they are just the parameters we wrote down. Once loops are included, the measured mass is some function of m, g and the loops, and there is no reason for it to equal m. So the right question is not "what is Π_{loop}?" but "what does an experiment actually see?". The plan:

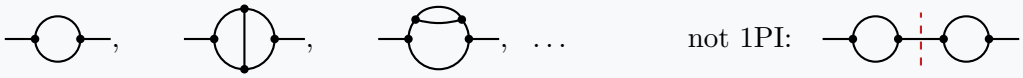
(i) work out what is measured, and turn it into conditions on the theory (Step 7);

(ii) give the Lagrangian enough freedom to satisfy them, and fix that freedom by the conditions (Step 8). The infinity will then disappear from every prediction.

Step 7 • Choose renormalization scheme

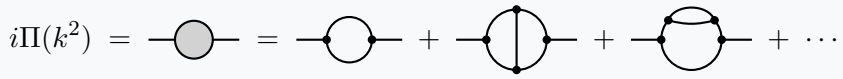
What does an experiment see? Particles, and their masses, show up in the **full propagator** of the interacting theory, the Fourier transform of $\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle$. It receives *all* loop contributions, not just our bubble; we draw it as a thick line, ━━━.

One-particle-irreducible diagrams. A diagram is *one-particle-irreducible* (1PI) if it stays connected when any single internal line is cut. For two external lines (removed),



(32)

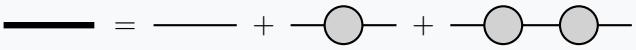
Call $i\Pi(k^2)$ the sum of *all* 1PI diagrams and draw it as a shaded blob:



(33)

So Π is not the same as Π_{loop}: the bubble Π_{loop} is the first term, of order g^2; the two-loop diagrams are of order g^4, and in Step 8 the counterterms will join in. For now, $\Pi = \Pi_{\text{loop}} + \mathcal{O}(g^4)$.

The full propagator. Cut every line that disconnects a diagram: what is left is a chain of 1PI pieces joined by single propagators. So every diagram for the full propagator is a chain of blobs, and each extra blob brings $i\Pi \times \frac{-i}{k^2+m^2} = \frac{\Pi}{k^2+m^2}$. The chain is a geometric series:



$$= \frac{-i}{k^2 + m^2 - i\varepsilon}\sum_{n=0}^{\infty}\left(\frac{\Pi(k^2)}{k^2 + m^2 - i\varepsilon}\right)^n = \frac{-i}{k^2 + m^2 - i\varepsilon - \Pi(k^2)}. \tag{34}$$

Two facts we know:

(1) A particle of mass m shows up as a pole of the full propagator at $k^2 = -m^2$, and the mass we measure is the position of that pole. We want the m in the Lagrangian to *be* this measured mass.

(2) LSZ, as derived in lecture, assumes that the field creates the one-particle state with unit amplitude, $\langle k|\phi(0)|\Omega\rangle = 1$. For the propagator this means that near the pole it looks exactly like the free one, $\approx -i/(k^2 + m^2)$, with coefficient 1.

To see what these demand of Π, Taylor-expand the denominator around $k^2 = -m^2$ (a prime means d/dk^2):

$$k^2 + m^2 - \Pi(k^2) = \underbrace{-\Pi(-m^2)}_{\text{must vanish by (1)}} + (k^2+m^2)\underbrace{[1-\Pi'(-m^2)]}_{\text{must equal 1 by (2)}} + \mathcal{O}((k^2+m^2)^2). \tag{35}$$

These are the **on-shell conditions**:

$$\Pi(-m^2) = 0, \qquad \Pi'(-m^2) = 0. \tag{36}$$

Back to our infinity. With the Lagrangian as written so far (all $Z = 1$), the only 1PI diagram at order g^2 is the bubble, $\Pi = \Pi_{\text{loop}} + \mathcal{O}(g^4)$. Setting $k^2 = -m^2$ in the pole part found in Step 6, and differentiating it,

$$\Pi_{\text{loop}}(-m^2) = -\frac{5g^2}{6(4\pi)^3\epsilon}\, m^2 + \text{finite}, \qquad \Pi'_{\text{loop}}(-m^2) = -\frac{g^2}{6(4\pi)^3\epsilon} + \text{finite}. \tag{37}$$

Both conditions are violated, and by infinite amounts. Now the infinity has a clear meaning: with all $Z = 1$ the Lagrangian has no freedom left to describe a particle of the measured mass m, normalized the way LSZ needs. To satisfy the two conditions we need two numbers in the Lagrangian that we are free to adjust, one for each condition, and the Lagrangian from lecture has exactly those.

Step 8 • Fix the counterterms

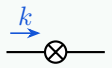
(a) The counterterms. The adjustable numbers are the Z's,

$$\mathcal{L} = -\tfrac{1}{2}Z_\phi\, \partial_\mu\phi\, \partial^\mu\phi - \tfrac{1}{2}Z_m m^2\phi^2 + \tfrac{1}{6}Z_g\, g\, \phi^3 + Y\phi, \qquad Z = 1 + Z^{(1)} + Z^{(2)} + \cdots, \tag{38}$$

where $Z^{(n)}$ is of order g^{2n}. The terms with $Z = 1$ are what we used so far. The terms with $Z_\phi^{(1)}$ and $Z_m^{(1)}$,

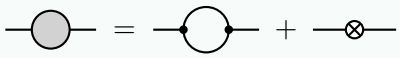
$$-\tfrac{1}{2}Z_\phi^{(1)}\, \partial_\mu\phi\, \partial^\mu\phi - \tfrac{1}{2}Z_m^{(1)}\, m^2\phi^2, \tag{39}$$

are new interactions with two legs, called *counterterms*. Their Feynman rule is



$$= -i(Z_\phi^{(1)}k^2 + Z_m^{(1)}m^2) \tag{40}$$

(the derivatives give $(ik)\cdot(-ik) = k^2$, and the $\frac{1}{2}$ cancels against the two ways of attaching the lines). The counterterm is of order g^2, like the bubble, so at this order the 1PI blob is



$$\Rightarrow \qquad \Pi(k^2) = \Pi_{\text{loop}}(k^2) - Z_\phi^{(1)}k^2 - Z_m^{(1)}m^2. \tag{41}$$

(b) Impose the conditions. Differentiating, $\Pi'(-m^2) = \Pi'_{\text{loop}}(-m^2) - Z_\phi^{(1)}$, and $\Pi(-m^2) = \Pi_{\text{loop}}(-m^2) + Z_\phi^{(1)}m^2 - Z_m^{(1)}m^2$. Setting both to zero,

$$Z_\phi^{(1)} = \Pi'_{\text{loop}}(-m^2), \qquad Z_m^{(1)} = Z_\phi^{(1)} + \frac{\Pi_{\text{loop}}(-m^2)}{m^2}. \tag{42}$$

(c) Evaluate them. Differentiate the result of Step 6 and integrate over x:

$$Z_\phi^{(1)} = -\frac{g^2}{6(4\pi)^3}\Big[\frac{1}{\epsilon} + \ln\frac{\mu}{m} + \kappa_\phi\Big], \qquad Z_m^{(1)} = -\frac{g^2}{(4\pi)^3}\Big[\frac{1}{\epsilon} + \ln\frac{\mu}{m} + \kappa_m\Big], \tag{43}$$

with the finite constants

$$\kappa_\phi \equiv -3\int_0^1 dx\, x(1-x)\ln(1-x+x^2) = \frac{17}{6} - \frac{\sqrt{3}\,\pi}{2}, \qquad \kappa_m \equiv \frac{5}{12} - \frac{1}{2}\int_0^1 dx\, \ln(1-x+x^2) = \frac{17}{12} - \frac{\pi}{2\sqrt{3}}. \tag{44}$$

(d) The pole is gone. The pole parts of the counterterms, $-\frac{g^2}{6(4\pi)^3\epsilon}$ and $-\frac{g^2}{(4\pi)^3\epsilon}$, cancel the pole of the loop

for *every* k^2, not only at $k^2 = -m^2$:

$$\Pi(k^2)\Big|_{\text{pole}} = -\frac{g^2}{(4\pi)^3\epsilon}\Big(\frac{k^2}{6} + m^2\Big) + \frac{g^2}{6(4\pi)^3\epsilon}\,k^2 + \frac{g^2}{(4\pi)^3\epsilon}\,m^2 = 0. \tag{45}$$

To get the finite remainder, insert (b) into Π:

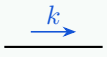
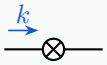
$$\Pi(k^2) = \Pi_{\text{loop}}(k^2) - \Pi_{\text{loop}}(-m^2) - (k^2 + m^2)\,\Pi'_{\text{loop}}(-m^2). \tag{46}$$

Therefore

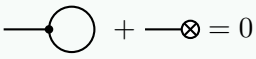
$$\Pi(k^2) = \frac{g^2}{2(4\pi)^3}\int_0^1 dx\; D\,\ln\frac{D}{D_0} \;-\; \frac{g^2}{12(4\pi)^3}(k^2 + m^2), \qquad \begin{aligned} D &= x(1-x)k^2 + m^2 - i\varepsilon, \\ D_0 &= m^2\big[1 - x(1-x)\big], \end{aligned} \tag{47}$$

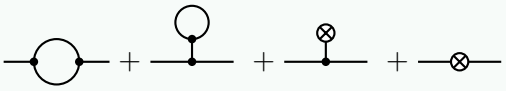
and the full propagator $-i/(k^2 + m^2 - i\varepsilon - \Pi(k^2))$ is finite. Note: no ϵ, no μ: only the measured mass m and the coupling g.

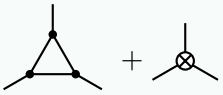
(e) Renormalized perturbation theory. What we just did is the general method, called **renormalized perturbation theory**: compute with the Lagrangian from lecture, with the *measured* m and g, and fix the Z's order by order by physical conditions. The rules:

term in $\mathcal{L}$	diagram	Feynman rule	fixed by
$-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$		$\dfrac{-i}{k^2 + m^2 - i\varepsilon}$	m = measured mass
$\frac{1}{6}\,g\,\phi^3$		ig	g = measured coupling
$-\frac{1}{2}Z_\phi^{(1)}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}Z_m^{(1)}m^2\phi^2$		$-i(Z_\phi^{(1)}k^2 + Z_m^{(1)}m^2)$	on-shell conditions
$\frac{1}{6}Z_g^{(1)}\,g\,\phi^3$		$iZ_g^{(1)}\,g$	condition defining g
$Y\phi$		iY	$\langle\Omega\|\phi\|\Omega\rangle = 0$

The counterterm vertices (crossed circles) are added to the loop diagrams order by order; $Z_\phi^{(1)}, Z_m^{(1)}, Z_g^{(1)}$ are of order g^2 and Y of order g. At one loop, the divergent diagrams are those with $E = 1, 2, 3$ external lines (one can check this by counting powers of the loop momentum, as for the bubble). With their counterterms:

$E = 1$:  fixes Y

$E = 2$:  fixes $Z_\phi^{(1)}$, $Z_m^{(1)}$

$E = 3$:  fixes $Z_g^{(1)}$

Was that cheating? No infinity was subtracted from anything measured. Write $\phi_0 = Z_\phi^{1/2}\phi$, $m_0^2 = Z_m m^2/Z_\phi$, $g_0 = Z_g g/Z_\phi^{3/2}$: the Lagrangian from lecture becomes $-\frac{1}{2}\partial_\mu\phi_0\,\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 + \frac{1}{6}g_0\phi_0^3 + \ldots$, with no Z's. So the infinities sit in the *bare* parameters m_0, g_0 and in the normalization of the bare field ϕ_0, which nobody measures. They blow up as $\epsilon \to 0$ and depend on the arbitrary μ, and that is fine: their job is to absorb the regulator so that Π does not have to.

5 Renormalization schemes

5.1 What is forced and what is a choice

Take $\Pi = \Pi_{\text{loop}} - Z_\phi^{(1)} k^2 - Z_m^{(1)} m^2$ with the ϵ-expanded loop. Whatever we do, the counterterms must cancel the pole, so their $1/\epsilon$ parts are forced:

$$Z_\phi^{(1)} = -\frac{g^2}{6(4\pi)^3\epsilon} + a_\phi, \qquad Z_m^{(1)} = -\frac{g^2}{(4\pi)^3\epsilon} + a_m, \tag{48}$$

with finite a_ϕ, a_m. Inserting this, the pole cancels and

$$\Pi(k^2) = -\frac{g^2}{2(4\pi)^3}\int_0^1 dx\, D\Big[1 + \ln\frac{\mu^2}{D}\Big] - a_\phi\, k^2 - a_m\, m^2. \tag{49}$$

The finite parts a_ϕ, a_m are *not* fixed by the requirement of finiteness. Choosing them is choosing a **renormalization scheme**. Different choices give different meanings to the symbols m and ϕ in the Lagrangian, but the same physics. Note the difference: a regulator is a way to make integrals finite; a scheme is a way to fix the finite parts of the counterterms. The on-shell scheme can be used with any regulator; MS and $\overline{\text{MS}}$ are defined through the $1/\epsilon$ poles, so they belong to dimensional regularization.

On-shell scheme (the one used so far)

Conditions: $\Pi(-m^2) = 0$, $\Pi'(-m^2) = 0$, imposed at the physical pole.

Counterterms:

$$a_\phi^{\text{OS}} = -\frac{g^2}{6(4\pi)^3}\Big[\ln\frac{\mu}{m} + \kappa_\phi\Big], \qquad a_m^{\text{OS}} = -\frac{g^2}{(4\pi)^3}\Big[\ln\frac{\mu}{m} + \kappa_m\Big]. \tag{50}$$

Meaning of the parameters: m is the pole mass; ϕ is normalized as LSZ assumes, so LSZ applies as it stands. m, ϕ and $\Pi(k^2)$ do not depend on μ; only the counterterms do.

Drawbacks: it needs an on-shell point. For massless particles, Π' at $k^2 = 0$ can be singular; for quarks, which never appear as free particles, the pole mass cannot be measured directly; for unstable particles the pole is complex. The algebra is also the heaviest of all schemes.

Minimal subtraction: MS and $\overline{\text{MS}}$

Rule: the counterterms remove the pole and nothing else, $a_\phi = a_m = 0$:

$$Z_\phi^{(1)}|_{\overline{\text{MS}}} = -\frac{g^2}{6(4\pi)^3\epsilon}, \qquad Z_m^{(1)}|_{\overline{\text{MS}}} = -\frac{g^2}{(4\pi)^3\epsilon}, \qquad \Pi_{\overline{\text{MS}}}(k^2) = -\frac{g^2}{2(4\pi)^3}\int_0^1 dx\, D\Big[1 + \ln\frac{\mu^2}{D}\Big], \tag{51}$$

with the Lagrangian mass, now called $\bar{m}$, inside D.

MS versus $\overline{\text{MS}}$. With the coupling written as $g\tilde{\mu}^{\epsilon/2}$, every one-loop pole comes in the same combination, $\frac{2}{\epsilon} - \gamma_E + \ln 4\pi$. There are two natural ways to subtract it:

- **MS** ("minimal subtraction") removes only $\frac{2}{\epsilon}$, so the leftover $-\gamma_E + \ln 4\pi$ appears in every finite answer;
- **$\overline{\text{MS}}$** removes the whole combination. Equivalently, write everything in terms of $\mu^2 = 4\pi e^{-\gamma_E}\tilde{\mu}^2$, as we did in Step 6: then the combination is just $\frac{2}{\epsilon} + \ln\mu^2$, and $\overline{\text{MS}}$ removes the pure pole $\frac{2}{\epsilon}$.

$\overline{\text{MS}}$ is the standard one because its formulas are the cleanest.

Meaning of the parameters: $\bar{m}$ is not the pole mass but a parameter $\bar{m}(\mu)$ that depends on the scale μ, and the field is not normalized as LSZ assumes.

Advantages: the counterterms are as simple as they can be (pure poles), they do not depend on the masses, and no on-shell point is needed. QCD results and the running couplings of the Standard Model are usually quoted in $\overline{\text{MS}}$.