Section 8: Groups, Representations and Lorentz Transformations

Wednesday, October 28th 2026

Lecture notes §7.1–7.2; Srednicki Ch. 2, 33, 34; Schwartz §10.1

Contents

1 Intro **1**

2 Groups and representations **1**

3 $SO(3)$ and its representations **4**

4 $SU(2)$, the double cover and spin **5**

5 The Lorentz group **7**

6 Representations of the Lorentz group **8**

1 Intro

So far every field in this course was a scalar: one component, unchanged by a Lorentz transformation, $\phi(x) \to \phi(\Lambda^{-1}x)$ under a Lorentz transformation $x \to \Lambda x$. Electrons and photons carry spin: their fields have several components, which a Lorentz transformation mixes,

$$\phi_a(x) \;\to\; D_{ab}(\Lambda)\,\phi_b(\Lambda^{-1}x). \tag{1}$$

Two transformations in a row must act like their product, $D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2)$. So "which fields can we write down?" becomes "which sets of matrices multiply like Lorentz transformations?", a question about *groups* and their *representations*.

2 Groups and representations

Everything about groups that we use today. If you have seen it before, skim it.

2.1 What a group is

A **group** is a set G together with a rule that combines any two elements $g_1, g_2 \in G$ into a product g_1g_2, such that

1. *Closure:* the product is again in the set, $g_1g_2 \in G$.
2. *Associativity:* $(g_1g_2)g_3 = g_1(g_2g_3)$.
3. *Identity:* there is an element 1 with $1g = g1 = g$ for every g.
4. *Inverse:* every g has an element g^{-1} with $gg^{-1} = g^{-1}g = 1$.

In physics a group is almost always a set of *transformations* of something. "Multiply" then means "do one transformation, then the other", the identity means "do nothing" and the inverse means "undo". Associativity is automatic for transformations, so the real content is closure: two allowed transformations in a row must again be an allowed transformation.

Examples and non-examples.

set	product	identity	inverse	group?
$\{+1,-1\}$	multiplication	$+1$	itself	yes: $\mathbb{Z}_2$ (parity)
phases $e^{i\alpha}$	multiplication	1	$e^{-i\alpha}$	yes: $U(1)$
rotations of space	one after the other	do nothing	rotate back	yes: $SO(3)$
invertible matrices	matrix product	$\mathbf{1}$	A^{-1}	yes: $GL(n)$
integers	multiplication	1	none for 2	no
odd integers	addition	(no 0)		no: odd + odd is even

Order matters. In general $g_1 g_2 \neq g_2 g_1$. Groups in which the order never matters ($\mathbb{Z}_2$, $U(1)$, the integers) are **abelian**, the others (like rotations) **non-abelian**. A **subgroup** is a subset that is a group by itself. For example, the rotations about the z-axis form a subgroup of all rotations, and $\{+1,-1\}$ is a subgroup of $U(1)$.

2.2 Matrix groups

Most groups we meet are sets of matrices, defined by some *condition* that survives multiplication. Usually the condition says which quantity the transformation keeps fixed.

group	matrices	condition, and what it keeps fixed
$O(n)$, $SO(n)$	real $n\times n$	$R^T R=\mathbf{1}$: lengths $\vec{x}^{\,T}\vec{x}$. $SO(n)$ also has $\det R=1$
$U(n)$, $SU(n)$	complex $n\times n$	$U^\dagger U=\mathbf{1}$: norms $\sum_a \lvert\psi_a\rvert^2$. $SU(n)$ also has $\det U=1$
$O(3,1)$	real 4×4	$\Lambda^T g\Lambda = g$, with g the metric: the interval $-t^2+\vec{x}^{\,2}$

The letters stand for "orthogonal", "unitary" and "special" (determinant one).

2.3 Lie groups and generators

All the groups in the table depend smoothly on continuous parameters. Such a group is called a **Lie group**. Close to the identity every element can be written as

$$g \approx \mathbf{1} - i\,\epsilon_a T_a, \tag{2}$$

where the ϵ_a are small numbers ($a=1,\dots,n$, summed), and the T_a are fixed matrices, the **generators**. The factor $-i$ is a convention chosen so that unitary groups have hermitian generators: $U^\dagger U \approx (\mathbf{1}+i\epsilon T^\dagger)(\mathbf{1}-i\epsilon T) \approx \mathbf{1}+i\epsilon(T^\dagger - T)$ equals $\mathbf{1}$ exactly when $T^\dagger = T$.

A finite transformation, with finite parameters θ_a, is many small steps in a row. Dividing each θ_a into N steps,

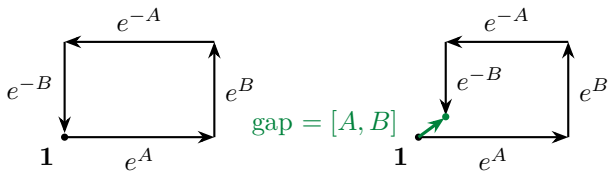
$$g = \lim_{N\to\infty}\Big(\mathbf{1}-\frac{i\theta_a T_a}{N}\Big)^N = e^{-i\theta_a T_a}, \qquad e^A \equiv \sum_{n=0}^{\infty}\frac{A^n}{n!}. \tag{3}$$

How the generators know the order matters. Take two small steps, $A=-i\epsilon T_a$ and $B=-i\epsilon T_b$, and go around a small "square": along A, along B, back along A, back along B. Expanding each exponential to second order, $e^{\pm A}\approx \mathbf{1}\pm A+\frac{1}{2}A^2$, gives

$$e^A e^B \approx \mathbf{1}+(A+B)+(AB+\tfrac{1}{2}A^2+\tfrac{1}{2}B^2), \qquad e^{-A}e^{-B}\approx \mathbf{1}-(A+B)+(AB+\tfrac{1}{2}A^2+\tfrac{1}{2}B^2). \tag{4}$$

Multiplying the two, the first-order terms cancel, and the second-order terms are $2AB+A^2+B^2-(A+B)^2 = AB-BA$:

$$e^A e^B e^{-A} e^{-B} = \mathbf{1}+[A,B]+\mathcal{O}(\epsilon^3) = \mathbf{1}-\epsilon^2\,[T_a,T_b]+\mathcal{O}(\epsilon^3), \qquad [A,B]\equiv AB-BA. \tag{5}$$



abelian: the square closes non-abelian: it misses by $[A, B]$

The result is a group element close to $\mathbf{1}$, so the gap must itself be built from the generators. That gives the defining relation

$$[T_a, T_b] = i\, f_{abc}\, T_c. \tag{6}$$

The numbers f_{abc} are the **structure constants**, and the generators with this commutation rule form the **Lie algebra** of the group. For an abelian group all $f_{abc} = 0$. The algebra determines the group near the identity, but not always globally: two different groups can share the same algebra.

A group is **compact** if its parameters live in a bounded region: angles, with θ and $\theta + 2\pi$ the same rotation, are bounded, while the parameter of a boost (the rapidity) runs over all real numbers. A group is **connected** if every element can be reached continuously from $\mathbf{1}$. $O(3)$ is not connected, since $\det R = \pm 1$ cannot jump from one value to the other, while $SO(3)$ is connected, and every element of $SO(3)$ is an exponential of generators.

2.4 Representations

A map $f : G \to H$ between groups that respects multiplication, $f(g_1 g_2) = f(g_1) f(g_2)$, is a **homomorphism**. If it is also one-to-one and onto, it is an **isomorphism**, $G \cong H$: the same group written differently. A **representation** of G is a homomorphism $D : G \to GL(n, \mathbb{C})$ into the invertible complex $n \times n$ matrices:

$$D(g_1)\, D(g_2) = D(g_1 g_2) \qquad \text{for all } g_1, g_2 \in G. \tag{7}$$

The number n is the **dimension** of the representation. If n is a finite number, the $D(g)$ are ordinary $n \times n$ matrices acting on n components, like the 3×3 rotation matrices acting on a vector: a **finite-dimensional** representation. The same definition works when the object has infinitely many components, for example a wave function $\psi(\vec{x})$, with one value at every point: a rotation acts as $\psi(\vec{x}) \to \psi(R^{-1}\vec{x})$, and $D(g)$ is then an operator on an infinite-dimensional space instead of a finite matrix. That is an **infinite-dimensional** representation.

A representation is a way to let the group act on an object Φ with n components, $\Phi \mapsto D(g)\Phi$, consistently: doing g_2 and then g_1 must be the same as doing $g_1 g_2$. The same group can act on many different objects, so it has many representations, of different sizes. For a Lie group it is enough to find $n \times n$ matrices with the same commutators as the generators, $[T_a, T_b] = i f_{abc} T_c$. Then $D(g) = e^{-i\theta_a T_a}$ multiplies the right way, at least near the identity.

Examples.

- The *trivial* representation, $D(g) = 1$ for every g: the object does not change, like a scalar.
- The *defining* representation of a matrix group: $D(g) = g$, the matrix itself. For the rotation group, the rotation matrix R acting on a vector $\vec{x} \mapsto R\vec{x}$ is the defining representation, with $n = 3$.

Vocabulary.

- **Equivalent** representations differ only by a change of basis, $D'(g) = S\, D(g)\, S^{-1}$ with the same matrix S for every g. On the generators, $T'_a = S T_a S^{-1}$. They describe the same object.
- **Reducible**: in some basis all $D(g)$ are block diagonal, $D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$, written $D = D_1 \oplus D_2$ (the "direct sum"). Example: rotations act on $(t, \vec{x})$ as $\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$, a scalar plus a vector. If no basis does

this, the representation is **irreducible**. Irreducible representations are the building blocks.

- **Tensor product**: if Φ transforms with D_1 and χ with D_2, then the products of their components, $\Phi_a \chi_b$ (all $n_1 n_2$ of them), also transform among themselves. These bigger matrices are written $D_1 \otimes D_2$. Example from quantum mechanics: two spins $\frac{1}{2}$ combine into spin 0 and spin 1, written $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.
- **Unitary**: a representation is called unitary if every $D(g)$ is a unitary matrix. Not every representation is unitary (the Lorentz group below gives examples), but quantum states need unitary ones, so that probabilities are kept.

Take-away. a group is a set of transformations that can be composed and undone. For Lie groups the generators and their commutators $[T_a, T_b] = i f_{abc} T_c$ carry the information. A representation is a set of matrices that multiply like the group elements: a way for the group to act on an object with n components.

3 $SO(3)$ and its representations

3.1 The group $SO(3)$

Definition. $SO(3)$ is the group of real 3×3 matrices R with

$$R^T R = \mathbf{1}, \qquad \det R = 1. \tag{8}$$

It acts on vectors, $\vec{x} \mapsto R\vec{x}$, and keeps $|\vec{x}|^2$ fixed, while $\det R = 1$ excludes reflections. Acting on $\vec{x}$ with the matrix itself is the *defining* representation, with three components.

What the matrices look like. A rotation is fixed by an axis and an angle, three parameters. About the coordinate axes:

$$R(\theta,\hat{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R(\theta,\hat{y}) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad R(\theta,\hat{z}) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

The generators. Near the identity, $R \approx \mathbf{1} - i\epsilon_k J_k$. Since R is real, the J_k are purely imaginary, and $R^T R \approx \mathbf{1} - i\epsilon_k (J_k^T + J_k) = \mathbf{1}$ requires $J_k^T = -J_k$: imaginary antisymmetric, hence hermitian. Such a 3×3 matrix has the form

$$\begin{pmatrix} 0 & -ic & ib \\ ic & 0 & -ia \\ -ib & ia & 0 \end{pmatrix} = a \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}}_{J_1} + b \underbrace{\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}}_{J_2} + c \underbrace{\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{J_3} \tag{10}$$

with three real numbers a, b, c. So these three matrices are a basis for the generators. In one formula, $(J_k)_{ij} = -i\varepsilon_{kij}$, with the totally antisymmetric symbol ε_{ijk} ($\varepsilon_{123} = 1$, a sign flip for each swap of two indices, zero if two indices are equal).

The algebra. Multiplying out, $J_1 J_2 - J_2 J_1 = \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) = i J_3$, and the other pairs follow by cyclic relabelling:

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \qquad \text{the commutators of angular momentum in quantum mechanics.} \tag{11}$$

The structure constants are $f_{ijk} = \varepsilon_{ijk}$.

Exponentiating.

$$R(\theta,\hat{n}) = e^{-i\theta\,\hat{n}\cdot\vec{J}}, \qquad \text{for example} \qquad e^{-i\theta J_3} = R(\theta,\hat{z}) \text{ of Eq. (9).} \tag{12}$$

3.2 Other representations: spin j

The defining representation is one set of matrices with $[J_i, J_j] = i\varepsilon_{ijk}J_k$, but not the only one: any three matrices with these commutators let rotations act on some object. Quantum mechanics already found all of them. Recall how: for angular-momentum operators with these commutators one builds states $|j, m\rangle$ with

$$\vec{J}^2|j,m\rangle = j(j+1)\,|j,m\rangle, \qquad J_3|j,m\rangle = m\,|j,m\rangle, \qquad m = -j, -j+1, \ldots, j, \tag{13}$$

and $J_\pm = J_1 \pm iJ_2$ raise and lower m by one. The ladder must stop at both ends, which allows only $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ The numbers $\langle j, m'|J_k|j, m\rangle$ form three $(2j+1)\times(2j+1)$ matrices with the right commutators. For each j this gives exactly one irreducible representation (up to a change of basis), the **spin-j representation**:

spin j	size	generators	what transforms
0	1×1	$J_i = 0$	scalar
$\frac{1}{2}$	2×2	$J_i = \frac{1}{2}\sigma_i$	spinor
1	3×3	$(J_k)_{ij} = -i\varepsilon_{kij}$ (in some basis)	vector
j	$(2j+1)\times(2j+1)$	$\langle j, m'|J_k|j, m\rangle$	

Whether all of these are representations of the group $SO(3)$, and not only of its algebra, is not obvious, because the algebra fixes the group only near the identity. Spin $\frac{1}{2}$ representation shows the problem: a rotation by 2π about z is $R = \mathbf{1}$, but with $J_3 = \frac{1}{2}\sigma_3$ one gets $e^{-2\pi i J_3} = \mathrm{diag}(e^{-i\pi}, e^{i\pi}) = -\mathbf{1}$, while a representation must send $\mathbf{1}$ to $\mathbf{1}$.

Is spin $\frac{1}{2}$ a representation of the rotation group at all?

4 $SU(2)$, the double cover and spin

4.1 The group $SU(2)$

To answer this question, we find the group that the 2×2 generators $\frac{1}{2}\sigma_i$ really generate.

Definition. $SU(2)$ is the group of complex 2×2 matrices U with

$$U^\dagger U = \mathbf{1}, \qquad \det U = 1. \tag{14}$$

It acts on two-component objects $\psi = (\psi_1, \psi_2)^T$, such as the spin states of an electron, and keeps $|\psi_1|^2 + |\psi_2|^2$ fixed.

The generators. Near the identity, $U \approx \mathbf{1} - i\epsilon_a T_a$. Unitarity requires hermitian T_a, and $\det U = 1$ requires $\mathrm{Tr}\, T_a = 0$, since $\det e^A = e^{\mathrm{Tr}\, A}$. A hermitian traceless 2×2 matrix has the form

$$\begin{pmatrix} c & d-ie \\ d+ie & -c \end{pmatrix} = d\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_1} + e\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_2} + c\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_3} \tag{15}$$

with three real numbers c, d, e. So the **Pauli matrices** are a basis for the generators, and we choose $T_a = \frac{1}{2}\sigma_a$. Three generators means three parameters, as for $SO(3)$.

The algebra. Multiplying two of them, for example $\sigma_1\sigma_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) = i\sigma_3$ and $\sigma_2\sigma_1 = -i\sigma_3$, and doing the same for the other pairs, gives

$$\sigma_i\sigma_j = \delta_{ij}\,\mathbf{1} + i\varepsilon_{ijk}\sigma_k, \tag{16}$$

so $\sigma_i^2 = \mathbf{1}$, different σ's anticommute, and $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$. Dividing by 4,

$$\left[\tfrac{1}{2}\sigma_i, \tfrac{1}{2}\sigma_j\right] = i\varepsilon_{ijk}\,\tfrac{1}{2}\sigma_k \qquad \text{exactly the } SO(3) \text{ algebra, } [J_i, J_j] = i\varepsilon_{ijk}J_k. \tag{17}$$

$SU(2)$ and $SO(3)$ have the same Lie algebra, and $\vec{J} = \frac{1}{2}\vec{\sigma}$ is the spin-$\frac{1}{2}$ representation from the table above.

Exponentiating.

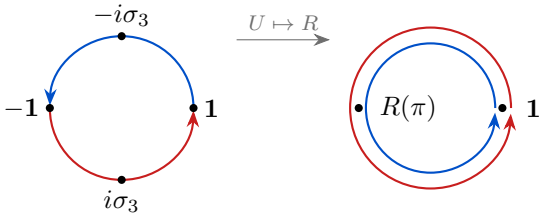
$$U(\theta, \hat{n}) = e^{-i\theta\,\hat{n}\cdot\vec{J}} = \cos\frac{\theta}{2}\,\mathbf{1} - i\,\hat{n}\cdot\vec{\sigma}\,\sin\frac{\theta}{2}, \qquad \text{for example} \qquad U(\theta, \hat{z}) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}. \tag{18}$$

4.2 $SU(2)$ and $SO(3)$: what is going on

Same algebra, different groups. Near the identity the two groups match element by element, so someone who only makes small rotations cannot tell them apart. They differ only *globally*. Rotate about z and let θ grow, in both groups at once:

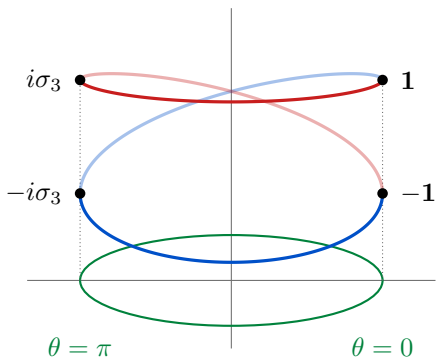
θ	0	π	2π	3π	4π
$SO(3)$: $R(\theta) = e^{-i\theta J_3}$	$\mathbf{1}$	$\mathrm{diag}(-1,-1,1)$	$\mathbf{1}$	$\mathrm{diag}(-1,-1,1)$	$\mathbf{1}$
$SU(2)$: $U(\theta) = e^{-i\theta\sigma_3/2}$	$\mathbf{1}$	$-i\sigma_3$	$-\mathbf{1}$	$i\sigma_3$	$\mathbf{1}$

When $R(\theta)$ has gone once around, $U(\theta)$ has only reached $-\mathbf{1}$. Send each $U(\theta)$ to the rotation $R(\theta)$ with the same angle: since $R(\theta + 2\pi) = R(\theta)$, the two matrices $U(\theta)$ and $U(\theta + 2\pi) = -U(\theta)$ land on the *same* rotation. The map is two-to-one, and both $\pm\mathbf{1}$ go to the identity rotation.

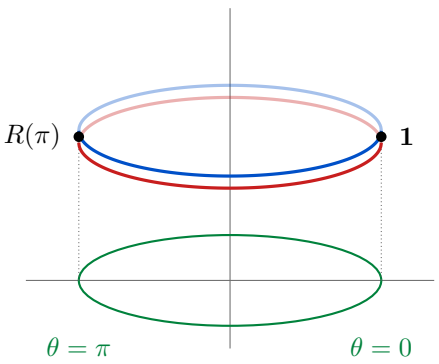


$SU(2)$ θ from 0 to 4π — $SO(3)$: rotations about z covered twice

blue: $\theta \in [0, 2\pi]$, red: $\theta \in [2\pi, 4\pi]$. Each half of the left circle covers the whole right circle.



$SU(2)$: $U(\theta)$ closes only after 4π



$SO(3)$: $R(\theta)$ closes after 2π, and is traced twice

The green circle is the rotation angle θ; above each point we draw the group element. Blue: first turn, $0 \le \theta \le 2\pi$; red: second turn.

What a covering is. A homomorphism $f : \tilde{G} \to G$ is a **covering** when it is onto, and one-to-one near the identity, so that $\tilde{G}$ and G have the same algebra. If every element of G comes from exactly two elements of a connected $\tilde{G}$, it is a **double cover**. So $SU(2)$ is the double cover of $SO(3)$, and $SO(3)$ is $SU(2)$ with U and $-U$ declared to be the same element.

What $Spin(d)$ means. The lecture uses the following name, for rotations in any number d of dimensions:

$Spin(d)$ is the (connected) double cover of $SO(d)$: the group with the same algebra as $SO(d)$, in which every rotation corresponds to two elements.

It is defined by this covering property, not by a matrix condition. For $d = 3$ we have just found it explicitly:

$Spin(3) \cong SU(2)$: whenever the lecture writes $Spin(3)$, read $SU(2)$.

4.3 Which spins belong to which group

Back to the question we asked about rotations: are the half-integer spins, such as spin $\frac{1}{2}$, representations of $SO(3)$? Every representation of $SO(3)$ gives one of $SU(2)$, $D(U) \equiv D(R)$, but a representation of $SU(2)$ comes from one of $SO(3)$ only if it treats U and $-U$ the same: $D(-\mathbf{1}) = \mathbf{1}$. In the spin-$j$ representation, a rotation by 2π multiplies the state with $J_3 = m$ by $e^{-2\pi i m} = (-1)^{2j}$, since $j - m$ is an integer. For integer j that is $+1$: honest representations of $SO(3)$. For half-integer j it is -1, so the answer to the question is *no*: half-integer spins are representations of $SU(2)$ but not of $SO(3)$, called **spinorial** representations.

spin	2π rotation	representation of	examples
$j = 0, 1, 2, \ldots$	$+1$	$SO(3)$ and $SU(2)$	scalars, vectors, orbital angular momentum
$j = \frac{1}{2}, \frac{3}{2}, \ldots$	-1	$SU(2)$ only	electron, proton, neutron

Nature is still allowed to use the spinorial ones, because a quantum state is defined only up to a phase: a sign under a 2π rotation changes no probability. The group acting on physical states is therefore $SU(2) = Spin(3)$, not $SO(3)$.

Take-away. $SU(2)$ has the same algebra as $SO(3)$, with half angles: every rotation comes from two matrices $\pm U$, so $SU(2) = Spin(3)$ is the double cover of $SO(3)$. Integer spins are representations of $SO(3)$, half-integer spins only of $SU(2)$, which is what nature uses.

5 The Lorentz group

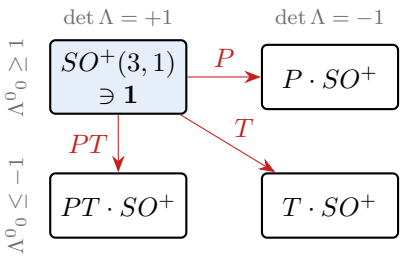
5.1 Definition

With the metric $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, Lorentz transformations $x \mapsto \Lambda x$ keep the interval $-t^2 + \vec{x}^2 = x^T g\, x$ fixed for every x, which means

$$\Lambda^T g\, \Lambda = g, \qquad \text{in indices} \qquad g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}. \tag{19}$$

These matrices form $O(3, 1)$, with "$(3, 1)$" for three plus signs and one minus sign in the metric.

Two consequences. Taking determinants, $(\det \Lambda)^2 = 1$, so $\det \Lambda = \pm 1$. The 00 component reads $-(\Lambda^0{}_0)^2 + \sum_i (\Lambda^i{}_0)^2 = -1$, so $(\Lambda^0{}_0)^2 \geq 1$: either $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$. Neither sign can change continuously, so $O(3, 1)$ falls into four pieces, connected by parity $P = \mathrm{diag}(1, -1, -1, -1)$ and time reversal $T = \mathrm{diag}(-1, 1, 1, 1)$. We only use the piece containing $\mathbf{1}$: the **proper** ($\det \Lambda = 1$) **orthochronous** ($\Lambda^0{}_0 \geq 1$) Lorentz group $SO^+(3, 1)$, made of all rotations and boosts.



5.2 Generators

Near the identity, $\Lambda = \mathbf{1}+\omega$ gives $\Lambda^T g\Lambda \approx g+\omega^T g+g\omega$, so $g\omega$ must be antisymmetric. An antisymmetric 4×4 matrix has $4\cdot 3/2 = 6$ entries, so there are six parameters: three angles $\vec{\theta}$ and three rapidities $\vec{\zeta}$. We write $\Lambda = e^{-i\vec{\theta}\cdot\vec{J}-i\vec{\zeta}\cdot\vec{K}}$. In the order (t,x,y,z), J_k is the 3×3 rotation generator placed in the spatial block, and the boost generators are

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad (K_i)^\mu{}_\nu = i(\delta^\mu_0\delta^i_\nu + \delta^\mu_i\delta^0_\nu), \qquad e^{-i\zeta K_3} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}. \tag{20}$$

Since ζ runs over all real numbers, the group is **non-compact**.

The algebra. The six generators obey

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \qquad [J_i, K_j] = i\varepsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\varepsilon_{ijk}J_k. \tag{21}$$

Boosts are not unitary. A rotation matrix is unitary: its entries are cosines and sines, never larger than one. A boost matrix is not: its entries γ and $\gamma\beta$ grow without bound as $v\to 1$, which no unitary matrix can do. In terms of generators, J_k is hermitian, so $e^{-i\theta J_k}$ is unitary, while K_i is anti-hermitian (i times a real symmetric matrix), so $e^{-i\zeta K_i}$ is not. This is not a defect: it is just that the Lorentz group has *no* finite-dimensional unitary representation other than the trivial one. It has many finite-dimensional representations (all of the next section), and it has unitary ones, but those are infinite-dimensional.

Take-away. $SO^+(3,1)$ is the piece of $\Lambda^T g\Lambda = g$ containing the identity: all rotations and boosts. Boosts are "rotations" with cosh and sinh, and two boosts in different directions leave a rotation. Boost matrices are not unitary.

6 Representations of the Lorentz group

6.1 Two copies of angular momentum

What are all Lorentz group representations? A representation is any set of matrices $\vec{J}, \vec{K}$ with the three commutators above. We know two already: the scalar ($\vec{J} = \vec{K} = 0$) and the four-vector (4×4). To find all of them, define

$$\vec{N} = \tfrac{1}{2}(\vec{J} - i\vec{K}), \qquad \vec{N}^\dagger = \tfrac{1}{2}(\vec{J} + i\vec{K}), \qquad \text{so that} \qquad \vec{J} = \vec{N} + \vec{N}^\dagger, \quad \vec{K} = i(\vec{N} - \vec{N}^\dagger). \tag{22}$$

The two copies decouple. The mixed commutator is

$$4[N_i, N_j^\dagger] = [J_i, J_j] + i[J_i, K_j] - i[K_i, J_j] + [K_i, K_j] = i\varepsilon_{ijk}J_k + i\cdot i\varepsilon_{ijk}K_k - i\cdot i\varepsilon_{ijk}K_k - i\varepsilon_{ijk}J_k = 0. \tag{23}$$

In the same way $[N_i, N_j] = i\varepsilon_{ijk}N_k$ and $[N_i^\dagger, N_j^\dagger] = i\varepsilon_{ijk}N_k^\dagger$. So the Lorentz algebra is *two independent copies of the angular-momentum algebra.* We know every representation of each copy: spin $0, \frac{1}{2}, 1, \ldots$ Hence

every irreducible finite-dimensional representation is labelled by two spins (j_1, j_2) and has $(2j_1+1)(2j_2+1)$ components.

What rotations see. $\vec{J} = \vec{N} + \vec{N}^\dagger$ is a sum of two commuting angular momenta. So if we only rotate, the components of (j_1, j_2) split into pieces of definite spin $|j_1 - j_2|, \ldots, j_1 + j_2$, each once, exactly as two spins combine in quantum mechanics. For example, a four-vector $V^\mu = (V^0, \vec{V})$ is $(\frac{1}{2}, \frac{1}{2})$, which contains spin $0\oplus 1$: under rotations V^0 does not change (spin 0) and $\vec{V}$ rotates as a vector (spin 1).

(j_1, j_2)	components	under rotations alone	sign at 2π	object	example
$(0,0)$	1	0	$+1$	scalar	Higgs field
$(\frac{1}{2},0)$, $(0,\frac{1}{2})$	2	$\frac{1}{2}$	-1	left-, right-handed spinor	neutrino field
$(\frac{1}{2},\frac{1}{2})$	4	$0\oplus 1$	$+1$	four-vector	x^μ, p^μ, A^μ
$(1,0)\oplus(0,1)$	6	$1\oplus 1$	$+1$	antisymmetric tensor	field strength $F^{\mu\nu}$: $\vec{E}$, $\vec{B}$
$(1,1)$	9	$0\oplus 1\oplus 2$	$+1$	symmetric traceless tensor	traceless stress tensor $T^{\mu\nu}$

The double cover again. When j_1+j_2 is half-integer there is a sign under a 2π rotation, so, exactly as for $SU(2)$ and $SO(3)$, those are representations only of the double cover of $SO^+(3,1)$, called $Spin^+(3,1)$. Lecture states that it is $SL(2,\mathbb{C})$, the complex 2×2 matrices with determinant one, by the same kind of accident as $Spin(3)\cong SU(2)$.

6.2 Complex conjugates

Complex fields come with their conjugates (ϕ and ϕ^*), so we need to know how conjugates transform. If the matrices $D(g)$ form a representation, so do their complex conjugates $D(g)^*$, since conjugation respects products. In terms of generators, $D=e^{-i\theta_a T_a}$ gives $D^*=e^{+i\theta_a T_a^*}=e^{-i\theta_a(-T_a^*)}$, so the conjugate representation has generators $-T_a^*$. Is it a new representation, or the old one in another basis? For spin $\frac{1}{2}$:

The conjugate generators are $-\frac{1}{2}\sigma_i^*$. We want to know whether a change of basis turns them back into $\frac{1}{2}\sigma_i$. Two steps:

- Conjugation: σ_1 and σ_3 are real and σ_2 is imaginary, so $-\sigma_i^*=(-\sigma_1,\ \sigma_2,\ -\sigma_3)$.
- Change of basis with σ_2: since σ_2 anticommutes with σ_1, σ_3 and squares to $\mathbf{1}$, $\sigma_2(\sigma_1,\sigma_2,\sigma_3)\sigma_2=(-\sigma_1,\ \sigma_2,\ -\sigma_3)$.

The two lists agree, so $-\sigma_i^*=\sigma_2\sigma_i\sigma_2$. Exponentiating, $U^*=\sigma_2 U\sigma_2=\varepsilon U\varepsilon^{-1}$ with $\varepsilon\equiv i\sigma_2=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$.

So for $SU(2)$ the conjugate of spin $\frac{1}{2}$ *is* spin $\frac{1}{2}$, only written in a basis rotated by σ_2, and the same holds for every spin j.

For the Lorentz group the conjugation is different: complex conjugation swaps the two spins, $(j_1,j_2)^*=(j_2,j_1)$. To see it, the conjugate representation has generators $\vec{J}'=-\vec{J}^*$ and $\vec{K}'=-\vec{K}^*$, so

$$\vec{N}'=\tfrac{1}{2}(\vec{J}'-i\vec{K}')=\tfrac{1}{2}(-\vec{J}^*+i\vec{K}^*)=-\Big[\tfrac{1}{2}(\vec{J}+i\vec{K})\Big]^*=-(\vec{N}^\dagger)^*,\qquad \vec{N}'^\dagger=-\vec{N}^*. \tag{24}$$

Complex conjugation turns $-i$ into $+i$ and so *swaps the roles* of $\vec{N}$ and $\vec{N}^\dagger$: $(j_1,j_2)^*=(j_2,j_1)$.

So, unlike for $SU(2)$, the two spinors $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ are genuinely different objects. Both rotate the same way, with $\vec{J}=\frac{1}{2}\vec{\sigma}$. They differ in how they boost: in one of them the second copy vanishes and $\vec{K}=+i\vec{J}$, in the other the first copy vanishes and $\vec{K}=-i\vec{J}$. They are called **left-** and **right-handed** spinors.

Take-away. the Lorentz algebra is two commuting copies of the angular-momentum algebra, $\vec{N}$ and $\vec{N}^\dagger$. Its representations are labelled (j_1,j_2), contain the spins $|j_1-j_2|,\dots,j_1+j_2$ under rotations, and complex conjugation and parity swap $j_1\leftrightarrow j_2$. The smallest nontrivial ones, $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, are the left- and right-handed spinors.