Section 9: The Lagrangian for Spin-$\frac{1}{2}$ Fields

Wednesday, November 4th 2026

Lecture notes §7.3–7.7; Srednicki Ch. 34–36; Schwartz Ch. 10; Dreiner, Haber, Martin, *Two-component spinor techniques*

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1 Intro

Recall the free scalar field. Its whole Lagrangian has two terms,

$$\mathcal{L} = \underbrace{-\tfrac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi}_{\text{kinetic term}}\ \underbrace{-\,\tfrac{1}{2}\,m^2\phi^2}_{\text{mass term}}. \tag{1}$$

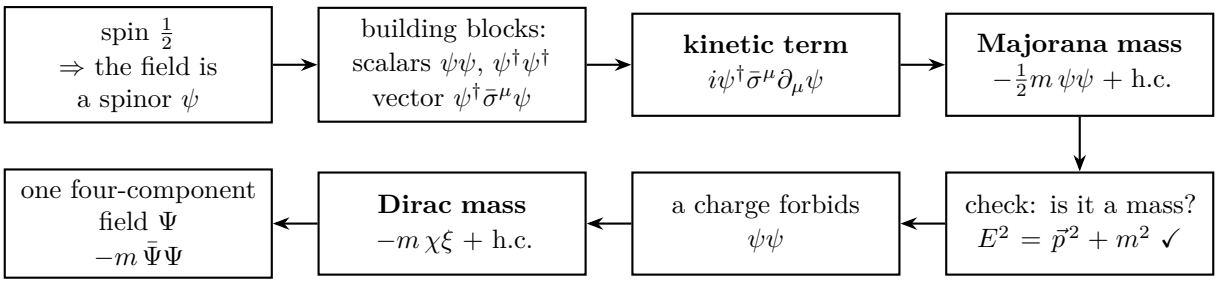
Electrons, quarks and neutrinos have spin $\frac{1}{2}$, and today we build the same two terms for a spin-$\frac{1}{2}$ field.

The goal for today. Build the free Lagrangian of a spin-$\frac{1}{2}$ field. Find all terms that are *quadratic in the field*, *Lorentz invariant* and *hermitian*, namely

- a **kinetic term**, with derivatives, so that the field propagates (like $-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$),
- a **mass term**, without derivatives (like $-\frac{1}{2}m^2\phi^2$),

and then check that the mass term really is a mass: it must give $E^2 = \vec{p}^{\,2} + m^2$.

Plan for today:



("h.c." stands for the hermitian conjugate of the term before it)

Why a spin-$\frac{1}{2}$ field is a spinor

Before we can write any term in the Lagrangian, we need to know what kind of object a spin-$\frac{1}{2}$ field is.

The field has to rotate like the particle. A spin-$\frac{1}{2}$ particle has two spin states, $|\uparrow\rangle$ and $|\downarrow\rangle$. From quantum mechanics we know that a rotation by an angle θ about the z-axis acts on them with the 2×2 matrix $e^{-i\theta\sigma_3/2}$. Take spin up, $|\uparrow\rangle = |\frac{1}{2},\frac{1}{2}\rangle$. Since $\sigma_3|\uparrow\rangle = |\uparrow\rangle$,

$$\text{state:} \qquad e^{-i\theta\sigma_3/2}\,|\uparrow\rangle = e^{-i\theta/2}\,|\uparrow\rangle \qquad \xrightarrow{\theta=2\pi} \qquad e^{-i\pi}\,|\uparrow\rangle = -\,|\uparrow\rangle. \tag{2}$$

After a full turn the *state* picks up a minus sign. Nothing measurable changes, because a state is only defined up to a phase, but the sign is there.

Now look at the field. A field acting on the vacuum creates the particle: $\phi_\uparrow(x)|0\rangle = |x,\uparrow\rangle$ is one particle at the point x with spin up. The vacuum looks the same from every direction, so rotating the state $|x,\uparrow\rangle$ is the same as rotating the field $\phi_\uparrow(x)$ in front of $|0\rangle$. After a full turn the point x is back where it was, and the state has picked up a minus sign, so the field must pick it up too:

$$\text{field:} \qquad \phi_\uparrow(x)|0\rangle = |x,\uparrow\rangle \xrightarrow{\text{full turn}} -|x,\uparrow\rangle \qquad \Longrightarrow \qquad \phi_\uparrow(x) \;\to\; -\phi_\uparrow(x). \tag{3}$$

A spin-$\frac{1}{2}$ field therefore needs two components, and it must flip sign after a full turn, exactly like the state it creates.

Which fields can do that? The fields we can write down are the representations of the Lorentz group:

Reminder: Lorentz representations

The rotation generators $\vec{J}$ and boost generators $\vec{K}$ combine into $\vec{N} = \frac{1}{2}(\vec{J} - i\vec{K})$ and $\vec{N}^\dagger = \frac{1}{2}(\vec{J} + i\vec{K})$, two commuting copies of the angular-momentum algebra. So every irreducible representation is labelled by two spins (j_1, j_2). The matrix that represents a full 2π turn is $+\mathbf{1}$ if $j_1 + j_2$ is an integer, so the field components come back to themselves, and $-\mathbf{1}$ if $j_1 + j_2$ is a half-integer, so all components flip sign.

(j_1, j_2)	sign at 2π	object
$(0,0)$	$+1$	scalar
$(\frac{1}{2},0)$, $(0,\frac{1}{2})$	-1	left-, right-handed spinor
$(\frac{1}{2},\frac{1}{2})$	$+1$	four-vector
$(1,0)\oplus(0,1)$	$+1$	antisymmetric tensor

The answer: spinors. Scalars, four-vectors and all tensors built from four-vectors are unchanged by a full turn, so none of them can describe a spin-$\frac{1}{2}$ particle. We need $j_1 + j_2$ to be a half-integer, and such representations are called **spinors**. The smallest ones are $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$.

2 Left- and right-handed spinors

The left-handed spinor. A left-handed spinor field $\psi_a(x)$, $a = 1, 2$, has two complex components, which we collect in a column. Under a Lorentz transformation the column is multiplied by a 2×2 matrix M:

$$\psi = \begin{array}{|c|}\hline \psi_1 \\ \hline \psi_2 \\ \hline \end{array} \quad \to \quad M\psi = \begin{array}{|c|}\hline M \\ \hline \end{array} \begin{array}{|c|}\hline \psi_1 \\ \hline \psi_2 \\ \hline \end{array}, \qquad M = \mathbf{1} - \tfrac{i}{2}\,\vec{\theta}\cdot\vec{\sigma} - \tfrac{1}{2}\,\vec{\zeta}\cdot\vec{\sigma} \quad \text{(infinitesimal)}, \tag{4}$$

with rotation angles $\vec{\theta}$ and rapidities $\vec{\zeta}$.

Its conjugate is right-handed. Now take the hermitian conjugate. As in quantum mechanics, the dagger of a column means conjugate *and* transpose, so $\psi^\dagger$ is a row. Its components are $\psi^\dagger_{\dot a} \equiv (\psi_a)^\dagger$: the dot does not modify the running of the indices over 1 and 2, it just notated that the object transforms with $M^\dagger$. Conjugating $\psi \to M\psi$ gives

$$\psi^\dagger = \begin{array}{|c|c|}\hline \psi^\dagger_{\dot 1} & \psi^\dagger_{\dot 2} \\ \hline \end{array} \quad \to \quad \psi^\dagger M^\dagger. \tag{5}$$

The conjugated components form the right-handed $(0, \frac{1}{2})$ spinor.

Raising indices with ε. To build Lorentz scalars we also need upper indices. We raise an index with the **antisymmetric symbol** ε^{ab}, whose entries are $\varepsilon^{12} = 1$, $\varepsilon^{21} = -1$ and $\varepsilon^{11} = \varepsilon^{22} = 0$:

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2, \qquad \varepsilon^T = -\varepsilon = \varepsilon^{-1}, \qquad \varepsilon^2 = -\mathbf{1}. \tag{6}$$

Raising the index, $\psi^a = \varepsilon^{ab}\psi_b$, swaps the two components and puts a minus sign on one of them, $\psi^1 = \psi_2$ and $\psi^2 = -\psi_1$. Dotted indices are raised with the same matrix, $\psi^{\dagger\dot a} = \varepsilon^{\dot a\dot b}\psi^\dagger_{\dot b}$, so $\psi^{\dagger\dot 1} = \psi^\dagger_{\dot 2}$ and $\psi^{\dagger\dot 2} = -\psi^\dagger_{\dot 1}$. For spinors, ε plays the role that the metric $g_{\mu\nu}$ plays for four-vectors. The metric turns two four-vectors into a scalar, $x^T g\, y$, because a Lorentz transformation $x \to \Lambda x$ obeys $\Lambda^T g \Lambda = g$. In the same way ε turns two spinors into a scalar, because $M^T \varepsilon M = \varepsilon$ for every Lorentz transformation M (see below).

Rows and columns. We use the convention of Dreiner, Haber and Martin: an upper index turns a column into a row and the other way around.

Left-handed $(\frac{1}{2}, 0)$

index	drawing	transforms as
ψ_a	$\begin{array}{\|c\|}\hline \psi_1 \\ \hline \psi_2 \\ \hline \end{array}$	$\to M\,(\cdot)$
ψ^a	$\begin{array}{\|c\|c\|}\hline \psi_2 & -\psi_1 \\ \hline \end{array}$	$\to (\cdot)\,M^{-1}$

Right-handed $(0, \frac{1}{2})$

index	drawing	transforms as
$\psi^\dagger_{\dot a}$	$\begin{array}{\|c\|c\|}\hline \psi^\dagger_{\dot 1} & \psi^\dagger_{\dot 2} \\ \hline \end{array}$	$\to (\cdot)\,M^\dagger$
$\psi^{\dagger\dot a}$	$\begin{array}{\|c\|}\hline \psi^\dagger_{\dot 2} \\ \hline -\psi^\dagger_{\dot 1} \\ \hline \end{array}$	$\to (M^\dagger)^{-1}(\cdot)$

The upper-index objects transform with the *inverse* matrices (this uses $M^T \varepsilon M = \varepsilon$).

3 Building blocks: which products are Lorentz invariant?

Every term in the Lagrangian is a product of two fields, possibly with derivatives. With ψ and $\psi^\dagger$ available there are three kinds of products: $\psi\psi$, $\psi^\dagger\psi^\dagger$ and $\psi^\dagger\psi$. How does each kind transform?

Looking at the representations. A product of two fields transforms in the product representation:

$$\underbrace{(\tfrac{1}{2},0)\otimes(\tfrac{1}{2},0)=(1,0)\oplus(0,0)}_{\psi\psi\text{: one scalar}},\qquad \underbrace{(0,\tfrac{1}{2})\otimes(0,\tfrac{1}{2})=(0,1)\oplus(0,0)}_{\psi^\dagger\psi^\dagger\text{: one scalar}},\qquad \underbrace{(0,\tfrac{1}{2})\otimes(\tfrac{1}{2},0)=(\tfrac{1}{2},\tfrac{1}{2})}_{\psi^\dagger\psi\text{: a four-vector, no scalar}}. \tag{7}$$

Each copy of spin combines as in quantum mechanics, $\frac{1}{2}\otimes\frac{1}{2}=1\oplus 0$. So we get two scalars and one four-vector. Let us build them.

The scalar from two left-handed spinors. For two left-handed spinors ψ and χ, we define

$$\psi\chi\equiv\psi^a\chi_a=\varepsilon^{ab}\psi_b\chi_a=\psi^T\varepsilon^T\chi=\begin{array}{|c|c|}\hline\psi_2&-\psi_1\\\hline\end{array}\begin{array}{|c|}\hline\chi_1\\\hline\chi_2\\\hline\end{array}=\psi_2\chi_1-\psi_1\chi_2. \tag{8}$$

In words, $\psi\chi$ is antisymmetric in the two spin indices, like the singlet $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle)$ in quantum mechanics. Under a Lorentz transformation both columns get multiplied by M, so

$$\psi\chi=\psi^T\varepsilon^T\chi\ \to\ (M\psi)^T\varepsilon^T(M\chi)=\psi^T(M^T\varepsilon^TM)\chi. \tag{9}$$

For any 2×2 matrix $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$,

$$M^T\varepsilon M=\begin{pmatrix}a&c\\b&d\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a&c\\b&d\end{pmatrix}\begin{pmatrix}c&d\\-a&-b\end{pmatrix}=\begin{pmatrix}0&ad-bc\\-(ad-bc)&0\end{pmatrix}=(\det M)\,\varepsilon. \tag{10}$$

The same holds for $\varepsilon^T=-\varepsilon$. Since $\det M=1$ (the Pauli matrices are traceless), $M^T\varepsilon^TM=\varepsilon^T$ and $\psi\chi$ is invariant.

The scalar from two right-handed spinors. Its partner is

$$\psi^\dagger\chi^\dagger\equiv\psi^\dagger_{\dot a}\chi^{\dagger\dot a}=\psi^\dagger_{\dot 1}\chi^\dagger_{\dot 2}-\psi^\dagger_{\dot 2}\chi^\dagger_{\dot 1}. \tag{11}$$

The row picks up $M^\dagger$ and the column $(M^\dagger)^{-1}$, so $\psi^\dagger\chi^\dagger\to\psi^\dagger M^\dagger(M^\dagger)^{-1}\chi^\dagger=\psi^\dagger\chi^\dagger$ is invariant too.

One of each: a four-vector, not a scalar. We can also write down $\psi^\dagger\psi=\psi^\dagger_1\psi_1+\psi^\dagger_2\psi_2$. It transforms as $\psi^\dagger\psi\to\psi^\dagger M^\dagger M\psi$.

- Rotation about z: $M=\mathbf{1}-\frac{i}{2}\theta\sigma_3$ is unitary to this order, $M^\dagger M=\mathbf{1}+\mathcal{O}(\theta^2)$, so $\psi^\dagger\psi$ is invariant under rotations.
- Boost along z: $M=\mathbf{1}-\frac{1}{2}\zeta\sigma_3$ is hermitian, so $M^\dagger M=\mathbf{1}-\zeta\sigma_3\neq\mathbf{1}$, so $\psi^\dagger\psi$ is *not* invariant under boosts.

The reason for such invariance is that $\psi^\dagger\psi$ is only one component of a four-vector. The rest of the components of the four-vector are built from

$$\sigma^\mu=(\mathbf{1},\ \vec\sigma),\qquad \bar\sigma^\mu=(\mathbf{1},\ -\vec\sigma),\qquad \mu=0,1,2,3, \tag{12}$$

four 2×2 matrices made of the identity and the three Pauli matrices $\sigma_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\sigma_2=\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)$, $\sigma_3=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. So once we sandwich the four-component sigma vector between $\psi^\dagger$ and ψ we get

$$\psi^\dagger\bar\sigma^\mu\psi\equiv\psi^\dagger_{\dot a}\,\bar\sigma^{\mu\dot a b}\,\psi_b=\begin{array}{|c|c|}\hline\psi^\dagger_{\dot 1}&\psi^\dagger_{\dot 2}\\\hline\end{array}\ \boxed{\bar\sigma^\mu}\ \begin{array}{|c|}\hline\psi_1\\\hline\psi_2\\\hline\end{array},\qquad \mu=0,1,2,3, \tag{13}$$

which transform as a four-vector. So since $\bar\sigma^0=\mathbf{1}$, the norm $\psi^\dagger\psi=\psi^\dagger\bar\sigma^0\psi$ is the time component, so it is invariant under rotations, but changes under boosts like the energy E.

product	components	transforms as
$\psi\psi = \psi^a\psi_a$	$\psi_2\psi_1 - \psi_1\psi_2$	scalar
$\psi^\dagger\psi^\dagger = \psi^\dagger_{\dot a}\psi^{\dagger\dot a}$	$\psi^\dagger_1\psi^\dagger_2 - \psi^\dagger_2\psi^\dagger_1$	scalar
$\psi^\dagger\bar\sigma^\mu\psi$	for $\mu = 0$: $\psi^\dagger_1\psi_1 + \psi^\dagger_2\psi_2$	four-vector

Note that the following expression from Eq. (8), $\psi\psi = \psi_2\psi_1 - \psi_1\psi_2$, would usually vanish for ordinary numbers. It does not vanish, because spinor fields *anticommute*:

$$\psi_a\chi_b = -\chi_b\psi_a, \qquad \text{in particular} \qquad \psi_2\psi_1 = -\psi_1\psi_2, \qquad \psi_1\psi_1 = \psi_2\psi_2 = 0, \tag{14}$$

while ordinary numbers commute with everything. That spinor fields anticommute is a consequence of the spin-statistics theorem. Schwartz (Chapter 12) gives a simple argument for it, and D. Harlow's QFT lecture notes give a more detailed mathematical proof.

4 The kinetic term

How many derivatives? For the scalar, $\partial_\mu\phi$ is a four-vector, and the only way to turn it into a scalar is to contract it with another four-vector built from ϕ, namely $\partial^\mu\phi$. That is why the scalar kinetic term has two derivatives. For a spinor there is a shortcut: the product $\psi^\dagger\bar\sigma^\mu\psi$ is *already* a four-vector, Eq. (13). Contracting it with a single derivative gives a scalar:

$$\psi^\dagger\bar\sigma^\mu\partial_\mu\psi = \begin{array}{|c|c|}\hline \psi_1^\dagger & \psi_2^\dagger \\ \hline \end{array} \quad \begin{array}{|c|}\hline \bar\sigma^\mu \\ \hline \end{array} \quad \begin{array}{|c|}\hline \partial_\mu\psi_1 \\ \hline \partial_\mu\psi_2 \\ \hline \end{array} \qquad \text{(summed over } \mu\text{)}. \tag{15}$$

Is it the only choice with one derivative? The derivative carries an index μ, so it must be contracted with a four-vector made from the two fields. The products $\psi\psi$ and $\psi^\dagger\psi^\dagger$ contain no four-vector, so $\psi^\dagger\bar\sigma^\mu\partial_\mu\psi$ is the only option. Putting the derivative on $\psi^\dagger$ instead gives minus the same term plus a total derivative, as we see now.

Making it hermitian. Take the hermitian conjugate of $X \equiv \psi^\dagger\bar\sigma^\mu\partial_\mu\psi = \psi^\dagger_{\dot a}\bar\sigma^{\mu\dot a b}\partial_\mu\psi_b$. The conjugate reverses the order and conjugates the matrix entries. Since $\bar\sigma^\mu$ is hermitian, $(\bar\sigma^{\mu\dot a b})^* = \bar\sigma^{\mu\dot b a}$, and

$$X^\dagger = (\partial_\mu\psi^\dagger_{\dot b})\,\bar\sigma^{\mu\dot b a}\,\psi_a = (\partial_\mu\psi^\dagger)\bar\sigma^\mu\psi = \underbrace{\partial_\mu(\psi^\dagger\bar\sigma^\mu\psi)}_{\text{total derivative}} - \psi^\dagger\bar\sigma^\mu\partial_\mu\psi. \tag{16}$$

A total derivative integrates to zero in the action, so $X^\dagger = -X$: the term is anti-hermitian. Multiplying by i fixes it, $(iX)^\dagger = -iX^\dagger = iX$. So the kinetic term is

$$\boxed{\mathcal{L}_{\text{kin}} = i\,\psi^\dagger\bar\sigma^\mu\partial_\mu\psi \equiv i\,\psi^\dagger_{\dot a}\,\bar\sigma^{\mu\dot a b}\,\partial_\mu\psi_b.} \tag{17}$$

5 The Majorana mass term

Now the term without derivatives. From the building blocks, the only Lorentz scalars made of two fields are $\psi\psi$ and $\psi^\dagger\psi^\dagger$. Take $\psi\psi$ with some coefficient m. The Lagrangian must be hermitian, and since the hermitian conjugate reverses the order of the two fields, $(m\,\psi\psi)^\dagger = m^*\psi^\dagger\psi^\dagger$, so we add both:

$$\mathcal{L}_{\text{Majorana}} = i\,\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - \tfrac{1}{2}\,m\,\psi\psi - \tfrac{1}{2}\,m^*\psi^\dagger\psi^\dagger. \tag{18}$$

The factor $\frac{1}{2}$ plays the same role as in $\frac{1}{2}m^2\phi^2$ for a real scalar: the term is quadratic in one field, and varying it gives a factor 2 that the $\frac{1}{2}$ cancels. The term is named after Ettore Majorana, who first studied fermions with this kind of mass in 1937.

For a single left-handed field, every term we are allowed to write is now in the Lagrangian. What is left is to show that m really is a mass.

Is m really a mass?

We want each component of ψ to obey the Klein–Gordon equation, so that a plane wave has $E^2 = \vec{p}^{\,2} + |m|^2$. Writing $\psi^\dagger\psi^\dagger = \psi^\dagger\varepsilon\psi^{\dagger T}$, where $\psi^{\dagger T}$ is the row $\psi^\dagger$ stood up as a column (so $\varepsilon\psi^{\dagger T}$ is the upper-index column from the right-handed table), and varying $\psi^\dagger$ in Eq. (18) gives the equation of motion. Its hermitian conjugate is the row equation $-i\,\partial_\mu\psi^\dagger\bar\sigma^\mu = m\,\psi^T\varepsilon^T$, and transposing it back into a column equation gives the second line:

$$i\,\bar\sigma^\mu\partial_\mu\psi = m^*\,\varepsilon\,\psi^{\dagger T}, \tag{19}$$

$$i\,(\bar\sigma^\mu)^T\partial_\mu\psi^{\dagger T} = -m\,\varepsilon\,\psi. \tag{20}$$

Compared with the massless case, the mass connects ψ to its own conjugate. We need two identities, both quick checks with Pauli matrices:

$$\text{(I)}\quad \sigma^\nu\bar\sigma^\mu + \sigma^\mu\bar\sigma^\nu = -2g^{\mu\nu}\,\mathbf{1}, \qquad g^{\mu\nu} = \mathrm{diag}(-1,1,1,1), \tag{21}$$

$$\text{(II)}\quad \sigma_2\,\sigma^\mu\,\sigma_2 = (\bar\sigma^\mu)^T, \qquad \text{and} \qquad \varepsilon^{-1}X\varepsilon = \sigma_2 X\sigma_2 \ \text{ for any matrix } X. \tag{22}$$

Now act with $i\sigma^\nu\partial_\nu$ on both sides of Eq. (19). The left side becomes

$$i\sigma^\nu\partial_\nu(i\bar\sigma^\mu\partial_\mu\psi) = -\sigma^\nu\bar\sigma^\mu\,\partial_\nu\partial_\mu\psi \overset{\partial_\nu\partial_\mu=\partial_\mu\partial_\nu}{=} -\tfrac{1}{2}(\sigma^\nu\bar\sigma^\mu + \sigma^\mu\bar\sigma^\nu)\partial_\nu\partial_\mu\psi \overset{\text{(I)}}{=} g^{\mu\nu}\partial_\mu\partial_\nu\psi = \partial^\mu\partial_\mu\psi. \tag{23}$$

The right side becomes

$$\begin{aligned}
i\sigma^\nu\partial_\nu(m^*\varepsilon\psi^{\dagger T}) &= m^*\,\varepsilon\,(\varepsilon^{-1}\sigma^\nu\varepsilon)\,i\partial_\nu\psi^{\dagger T} && \text{insert } \varepsilon\varepsilon^{-1} = \mathbf{1}\\
&= m^*\,\varepsilon\,(i\,(\bar\sigma^\nu)^T\partial_\nu\psi^{\dagger T}) && \text{(II): } \varepsilon^{-1}\sigma^\nu\varepsilon = \sigma_2\sigma^\nu\sigma_2 = (\bar\sigma^\nu)^T\\
&= m^*\,\varepsilon\,(-m\,\varepsilon\psi) && \text{Eq. (20)}\\
&= -|m|^2\,\varepsilon^2\,\psi = |m|^2\,\psi && \varepsilon^2 = -\mathbf{1}.
\end{aligned}$$

Setting the two sides equal:

$$\partial^\mu\partial_\mu\psi = |m|^2\psi \qquad\Longleftrightarrow\qquad (-\partial^\mu\partial_\mu + |m|^2)\psi = 0. \tag{24}$$

We get exactly the Klein–Gordon equation for each of the two components, so a plane wave $e^{ip\cdot x}$ needs $E^2 = \vec{p}^{\,2} + |m|^2$: the particle has mass $|m|$.

6 Charged fields: the Dirac mass

So far we only asked for Lorentz invariance, but now suppose the particle carries a charge.

Recall the complex scalar. The Lagrangian $\mathcal{L} = -\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$ is invariant under the phase rotation $\phi \to e^{i\alpha}\phi$, $\phi^* \to e^{-i\alpha}\phi^*$. By Noether's theorem this symmetry gives a conserved charge: the particle has charge $+1$ and the antiparticle -1. Notice that the mass term pairs ϕ with ϕ^*. A term like $m^2\phi\phi$ would turn into $e^{2i\alpha}m^2\phi\phi$ and break the symmetry.

The same test for the Majorana mass. Rotate $\psi \to e^{i\alpha}\psi$, so $\psi^\dagger \to e^{-i\alpha}\psi^\dagger$. Then

$$i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi \;\to\; e^{-i\alpha}e^{i\alpha}\, i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi \quad \text{(invariant)}, \qquad m\,\psi\psi \;\to\; e^{2i\alpha}\, m\,\psi\psi \quad \text{(not invariant)}. \tag{25}$$

The Majorana mass pairs ψ with ψ, like $\phi\phi$, so it breaks the symmetry. A charged particle such as the electron *cannot* have a Majorana mass. Without such a symmetry, there is no charge that tells a particle from its antiparticle: a Majorana fermion is its own antiparticle, just like the particle of a real scalar field. Neutrinos have no electric charge, so they may have Majorana masses (nobody knows yet!).

The fix: two fields with opposite charges. Take two left-handed fields χ and ξ with opposite charges,

$$\chi \to e^{i\alpha}\chi, \qquad \xi \to e^{-i\alpha}\xi. \tag{26}$$

Both kinetic terms are invariant. Every product of two left-handed fields is Lorentz invariant, so check the charges:

term	under the phase rotation	allowed?
$\chi\chi$	$e^{2i\alpha}\chi\chi$	no
$\xi\xi$	$e^{-2i\alpha}\xi\xi$	no
$\chi\xi$	$e^{i\alpha}e^{-i\alpha}\chi\xi = \chi\xi$	yes

Exactly one term survives, together with its hermitian conjugate $(\chi\xi)^\dagger = \xi^\dagger\chi^\dagger$. (Terms like $\chi^\dagger\xi$ are not Lorentz invariant, for the same reason as $\psi^\dagger\psi$.) The result is the Dirac Lagrangian:

$$\mathcal{L}_{\text{Dirac}} = i\,\chi^\dagger\bar\sigma^\mu\partial_\mu\chi + i\,\xi^\dagger\bar\sigma^\mu\partial_\mu\xi - m\,\chi\xi - m^*\,\xi^\dagger\chi^\dagger. \tag{27}$$

The phase of m can be removed by multiplying χ by a constant phase (which leaves the kinetic term unchanged), so from now on m is real. There is no $\frac{1}{2}$, as for the complex scalar: $\chi\xi$ contains two different fields, so varying it gives no factor 2.

What the two fields are, for the electron. The field χ carries the electron's charge. The field ξ carries the opposite charge, so its conjugate $\xi^\dagger$ carries the electron's charge again, and $\xi^\dagger$ is right-handed. So the electron is described by a left-handed part χ and a right-handed part $\xi^\dagger$, both with the same charge, and the mass term is what couples the two.

7 One four-component field

The Lagrangian in Eq. (27) works, but it has two fields that are always coupled by the mass. We can stack them into one column with four components:

$$\Psi = \begin{pmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{pmatrix} \begin{matrix} \leftarrow \text{left-handed} \\ \leftarrow \text{right-handed} \end{matrix} \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \bar\Psi \equiv \Psi^\dagger\beta, \tag{28}$$

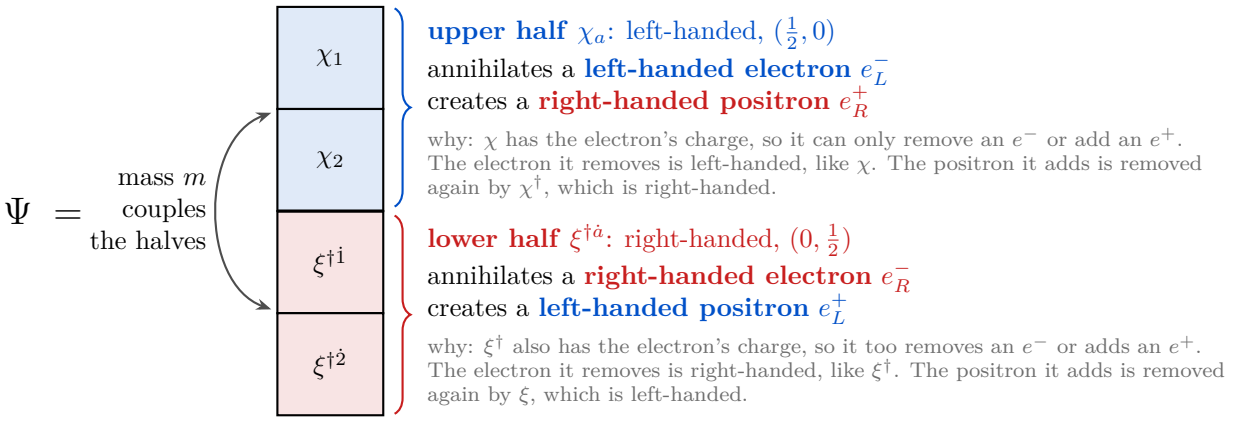
$$\mathcal{L}_{\text{Dirac}} = i\,\bar\Psi\gamma^\mu\partial_\mu\Psi - m\,\bar\Psi\Psi. \tag{29}$$

The gamma matrices obey the **Clifford algebra**

$$\{\gamma^\mu,\gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}\,\mathbf{1}. \tag{30}$$

To check it, multiply the blocks, $\gamma^\mu\gamma^\nu = \left(\begin{smallmatrix} \sigma^\mu\bar\sigma^\nu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu \end{smallmatrix}\right)$, and use identity (I) in each block (for the lower block the same check works with σ and $\bar\sigma$ swapped).

Here Ψ, the electron field, transforms in the $(\frac12,0)\oplus(0,\frac12)$ representation. Both of its halves carry the same charge, so the phase rotation is simply $\Psi \to e^{i\alpha}\Psi$, with $\bar\Psi \to e^{-i\alpha}\bar\Psi$, and Eq. (29) is invariant. The 4×4 matrices γ^μ are the gamma matrices, and $\bar\Psi$ is the Dirac conjugate. In our convention χ_a and $\xi^{\dagger\dot a}$ are both columns, so Ψ is a column, and $\bar\Psi = \Psi^\dagger\beta = (\xi^a,\ \chi^\dagger_{\dot a})$ is a row.



Both halves carry the electron's charge, so each half annihilates electrons and creates positrons.
The mass couples the halves, so a massive electron is a mix of e^-_L and e^-_R.

The Dirac equation. Treat Ψ and $\bar\Psi$ as independent, as we did for ψ and $\psi^\dagger$, and vary $\bar\Psi$ in Eq. (29). It appears on the left of both terms, so

$$\delta\mathcal{L} = \delta\bar\Psi(i\gamma^\mu\partial_\mu\Psi - m\Psi) \qquad\Longrightarrow\qquad (i\gamma^\mu\partial_\mu - m)\Psi = 0. \tag{31}$$

The result is the **Dirac equation**, which Paul Dirac wrote down for the electron in 1928. Multiplying out the blocks, $\gamma^\mu\partial_\mu\Psi = \left(\begin{smallmatrix} \sigma^\mu\partial_\mu\xi^\dagger \\ \bar\sigma^\mu\partial_\mu\chi \end{smallmatrix}\right)$, so it reads

$$i\,\sigma^\mu\partial_\mu\xi^\dagger = m\,\chi, \qquad\qquad i\,\bar\sigma^\mu\partial_\mu\chi = m\,\xi^\dagger. \tag{32}$$

The mass couples the left-handed χ to the right-handed $\xi^\dagger$, just as the Majorana mass coupled ψ to its own conjugate.

It is a mass. Act with $i\gamma^\nu\partial_\nu$ on $i\gamma^\mu\partial_\mu\Psi = m\Psi$. The right side gives $m\,i\gamma^\nu\partial_\nu\Psi = m^2\Psi$. On the left we use the Clifford algebra, Eq. (30):

$$i\gamma^\nu\partial_\nu(i\gamma^\mu\partial_\mu\Psi) = -\tfrac12(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)\partial_\nu\partial_\mu\Psi = \partial^\rho\partial_\rho\Psi = m^2\Psi, \tag{33}$$

the Klein–Gordon equation again, now with mass m for all four components.